Modelling and calibration of the Andrews-Mika diagram with a texture-dependent liberation model

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Abstract. Mineral liberation modelling has advanced from simple geometrical approaches to frameworks capable of describing ore texture variability and breakage-liberation phenomena. However, current models still present important limitations, either due to the large experimental effort required for calibration or the lack of direct coupling between texture, phase conservation, and breakage mechanics. This study uses a texture-dependent liberation model (TLM) in virtual experiments in calibration of an Andrews–Mika diagram model for a conceptual binary copper ore. Calibration of the Andrews-Mika diagram model showed that some of the model parameters vary significantly with particle grade, contrasting with the assumption that they were constant in the original formulation. The proposed approach bridges the gap between realistic TLMs and practical and phenomenological comminution models, thereby facilitating the integration of comminution and liberation models for full circuit simulations.

1. Introduction

Mineral liberation modelling has evolved considerably over the last century, progressing from purely geometrical approaches [1] to advanced frameworks incorporating mineral texture and breakage mechanisms [2-5].

Early liberation models aimed to describe the three-dimensional geometry of intact particles and predict the distribution of liberated and middling particles after breakage, although without a clear representation of the coupling between fragmentation and liberation phenomena [1]. While these approaches represented important advances in the phenomenological understanding of liberation, their application in comminution modelling remained limited. Significant progress was achieved in the early 1970s with the introduction of the Andrews–Mika diagram, a two-dimensional representation capable of mapping the distribution of progeny particles in the size-by-grade phase space after breakage [2]. Despite its relevance, the practical application of the diagram in comminution models was restricted by the complexity associated with its mathematical description [6].

A major advancement was only achieved in the late 1990s with the work of King and Schneider [6-7], who proposed a quadrivariate function capable of fully describing the Andrews–Mika diagram. This phenomenological framework provides a direct and practical representation of the diagram by separately predicting its boundaries, the internal distribution of middling particles, and the liberated fractions represented by Dirac functions at the extremes. This model directly interfaces with population balance formulations, enabling the coupled simulation of breakage and liberation.

Despite its advantages, the King and Schneider model requires a large amount of experimental data for calibration [6-7]. In parallel, geometric-textural approaches have also been proposed to explicitly represent mineral textures. Among these, the stochastic texture-dependent liberation model (TLM) proposed by Hilden and Powell [5] stands out by generating abstract intact ore textures calibrated from mineralogical image analysis. This approach enables the prediction of liberation spectra under different textural conditions and provides a realistic representation of texture variability effects. However, the model does not explicitly account for particle breakage mechanics and cannot meaningfully be incorporated in process simulators with recycle streams.

The present study considers a conceptual copper ore texture and applies the TLM [5] to perform virtual experiments aimed at supporting the calibration of the Andrews–Mika diagram model.

2. Modelling overview

2.1. Andrews-Mika diagram

The Andrews-Mika diagram is a graphical representation of the phase space available to progeny particles produced when a single parent particle breaks during comminution. The diagram is constructed in a two-dimensional space whose axes are particle size and grade. Any progeny particle in a population can be located as a point in this space and the diagram addresses where a fragment from a

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broken parent particle at a known position (grade g', size d_p') realistically appears in the phase space [2].

The key insight is that the progeny population is not free to occupy the entire phase space below the parent. Instead, it is strictly constrained by the conservation of mineral and gangue volumes. Because no progeny particle can carry more mineral than the mineral present in the parent and, equally, no progeny can carry more gangue than the gangue present in the parent, these two conservation requirements define jointly a bounded sub-region of the phase space. This sub-region of the phase space, called attainable region, has been modelled using a quadrivariate function that varies with grade and size of both parent and progeny particles [6-7].

A discretized model for the attainable region was proposed as [6-7]:

$$p_i\left(g|d_p; g', d_p'\right) = L_m\left[I_{\xi(g_{i+1})}(\alpha^M, \beta^M) - I_{\xi(g_i)}(\alpha^M, \beta^M)\right] \quad (1)$$

where g and d_p are the grade and size of the progeny particle, g' and d_p' are the grade and size of the parent particle, $\xi(g_i)$ the transformed grade variable and α^M and β^M are parameters of the incomplete beta function I.

Application of Eq. (1) requires: (1) calculating the boundaries of the phase-space; (2) defining the inner distribution based on the fraction of middling particles; (3) calculating the fraction of liberated mineral (L_1) and gangue (L_0). The upper (g_u) and lower (g_l) bounds of the Andrews-Mika diagram are calculated as:

$$g_u \leq max\left[g'\left(\frac{d_p'}{d_p}\right)^{\delta_{u,1}}, g'\left(\frac{D_{lib}}{d_p}\right)^{\delta_u}\right]$$
$$1 - g_l \leq max\left[(1-g')\left(\frac{d_p'}{d_p}\right)^{\delta_{l,1}}, (1-g')\left(\frac{D_{lib}}{d_p}\right)^{\delta_l}\right] \quad (2)$$

where δ_u, $\delta_{u,0}$, δ_l and $\delta_{l,0}$ are fitting parameters, while D_{lib} is the size where liberation starts to become statiscally relevant. These parameters indirectly capture the relationship between grain and particle size in the liberation behaviour. King [6] suggested $\delta_{u,1}$ and $\delta_{l,1}$ with constant value as 0.2, while δ_u and δ_l are predicted as:

$$\delta = min\left[\delta_0\left(\frac{D_{lib}}{d_p'}\right)^{\frac{1}{2}}, 3\right] \quad (3)$$

where δ_0 is a fitting parameter.

After properly defining the boundaries of the Andrews-Mika diagram (Eqns. (2) and (3)), the inner distribution may be defined by the incomplete beta function, shown in Eq. (1). At this stage it is worth noting that the inner distribution will extend only from g_l to g_u, which may not consider the entire range from 0 to 1. This situation is handled by assuming the transformed variable $\chi = \frac{g-g_l}{g_u-g_l}$ as an input to the model [6-7].

Parameters α^M and β^M (Eq. (1)) must be chosen to ensure that the average grade of the inner distribution ($\bar{g}^M$) matches the grade of the parent particle as:

$$\alpha^M = \bar{g}^M\gamma$$
$$\beta^M = (1-\bar{g}^M)\gamma \quad (4)$$

where γ is the standard deviation of the distribution given by $\gamma = \frac{1-f(d_p)}{f(d_p)}$. The term $f(d_p)$ is given as:

$$f(d_p) = \frac{1}{1 + \left(\frac{d_p}{D_{lib}}\right)^{\lambda}} \quad (5)$$

where λ is a fitting parameter.

The average grade of the inner distribution is then given as [6-7]:

$$\bar{g}^M = \int_{g_l}^{g_u} g p\left(g|d_p; g', d_p'\right) dg = \frac{g' - L_1}{1 - L_1 - L_0} \quad (6)$$

Eq. (7) relies on the fraction of liberated mineral (L_1) and gangue (L_0) in each size class. These fractions are also predicted using an incomplete beta function defined at the boundaries and based on the transformed variables as:

$$L_0 = I\left(\frac{-g_l}{g_u - g_l}, g'\gamma, (1-g')\gamma\right)$$
$$L_1 = 1 - I\left(\frac{1-g_l}{g_u - g_l}, g'\gamma, (1-g')\gamma\right) \quad (7)$$

2.2. Texture-dependent liberation model (TLM)

TLMs have been investigated by different authors since the early 2000s [4-5]. These frameworks combine detailed image analysis of the unbroken ore to characterize and model the ore texture, thus aiming to improve the prediction of liberation behaviour.

The TLM, proposed by Hilden and Powell [5], creates an abstract intact texture to capture the liberation distribution and properties of the fragmented particles. The model is designed to handle complex ores with different grades and grain size distributions, where the volume texture is characterized by [5]: phase type, modal mineralogy, grain shape, grain size distribution and spatial distribution.

The model considers particle shapes as spheres or orthogonally oriented cubes, while the grain size distribution is described using the Rosin-Rammler distribution [5]. The spatial distribution of grains within the parent particle is modelled based on the degree of clustering in the different axes, where the distribution considered is defined only by a filling factor [5]. The process of populating the texture volume involves randomly inserting primary grains into a 3D space using a random number generator. Progeny fragments are formed by sampling the texture volume to generate simulated particles in 2D or 3D, spanning various size classes. The breakage process assumes independence from grain boundaries or mineral properties, thus describing only random and non-selective breakage.

This modelling framework does not mechanically describe particle breakage. Instead, it considers the intact texture as having an infinite size and randomly selects samples within it for each progeny size. Initially, the progeny particle size to be sampled is defined. The model then uses a Monte Carlo algorithm to randomly select a coordinate within the intact texture, where a cube with an edge length equal to the progeny size and centred at this coordinate is generated. This cube represents a progeny particle of the selected size, and its composition is subsequently calculated. This procedure is typically repeated 10,000 times per size class to reliably estimate the liberation distribution for the given progeny size [5].

3. Materials and methods

3.1. Conceptual ore texture

The present work investigates the liberation behaviour of a hypothetical bornite copper ore. The ore texture is assumed to be a binary mixture of bornite and gangue, with specific gravities of 5.1 and 2.7 t/m³, respectively. 80% of the grains are finer than 1 mm, while the Rosin-Rammler slope of the distribution was equal to 2. As such, the minimum and maximum grain sizes varied from 16 μm to 2 mm, respectively. The texture filling parameter (Section 2.2) is assumed to be equal to 1.

3.2. Virtual experiments

Virtual experiments are then conducted using the TLM [5] discussed in Section 2.2, and the ore texture presented in Section 3.1.

These virtual experiments were initially performed considering the ore texture parameters as presented in Section 3.1 and assuming a texture size of 10 mm. A total of thirteen simulations were performed by varying the grade of the parent particle from 5 to 65% by volume. Since the model does not allow grains to overlap, the grade of 65% by volume, which represented 77.8% by weight, was the maximum achieved given the packing limit [5]. All simulations were carried out by considering progeny sizes varying from 8 mm to 0.29 μm. Parent particle size was assumed as 50 mm in these simulations.

3.3. Calibration of the Andrews-Mika diagram

From the virtual experiments described in Section 3.2, parameters in the attainable region equations presented in Section 2.1 were calibrated. In particular, the following model parameters were fitted: D_{lib}, $\delta_{0,u}$, $\delta_{0,l}$, $\delta_{l,1}$, $\delta_{u,1}$ and λ. Parameters $\delta_{0,u}$ and $\delta_{0,l}$ from Eq. (3) may assume different values for the upper and lower boundaries of the Andrews–Mika diagram.

The parameters are fitted separately by first describing the boundaries of the Andrews–Mika diagram using Eqns. (2) and (3), which depend on D_{lib}, $\delta_{0,u}$, $\delta_{0,l}$, $\delta_{l,1}$ and $\delta_{u,1}$. The remaining parameter, λ, is subsequently fitted using the extracted values of the liberated ore mineral and gangue fractions (L_1 and L_0) through Eq. (7). The optimal parameter estimation is carried out using a nonlinear optimization method based on Sequential Quadratic Programming.

4. Results and discussions

Figure 1 presents the 3D liberation spectrum predicted using the TLM for a particle with 20% grade by volume.

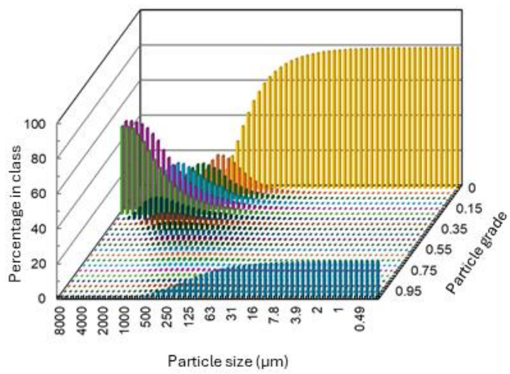

Figure 1. 3D liberation spectrum predicted using the TLM from breakage of a 20% grade by volume particle.

Data from Figure 1, which was generated by simulating a texture with 80% of the grains finer than 1 mm, is presented in Figure 2 to construct the boundaries of the Andrews-Mika diagram. Markers in Figure 2 represent the virtual experiment predicted with the TLM and lines the fitting of Eqns. (2) and (3). Good agreement is observed between them, with optimal values of parameters $\delta_{0,u}$ as 2.3, $\delta_{0,l}$ as 0.7, $\delta_{u,1}$ as 0.03, $\delta_{l,1}$ as 0.073 and D_{lib} as 6 mm.

Figure 2 also presents a comparison between the boundaries of the Andrews-Mika diagram by considering textures with different grain size distributions. Interestingly, results were well normalized in respect to grain size, where d_g in Figure 2 represents the 80% passing size of the grain size distribution.

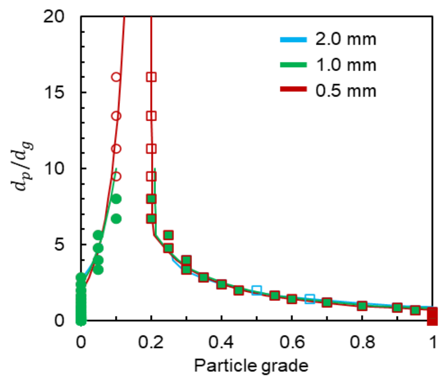

Figure 2. Comparison between the boundaries of the Andrews-Mika diagram predicted by the TLM and fitted with Eqns. (2) and (3) for a parent particle with 20% grade by volume and different grain size distributions.

The model that describes the attainable region [6-7], presented in Section 2.1, assumes that parameters from Eqns. (2), (3) and (6) must remain constant in respect to the parent grade, being then assumed as global ore texture parameters. However, results in Figure 3 demonstrate a strong dependence of $\delta_{0,u}$ and $\delta_{0,l}$ with the parent grade.

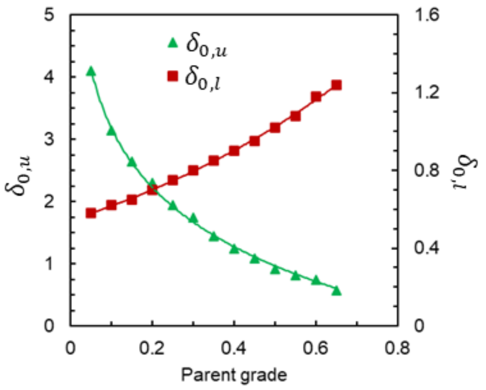

Figure 3. Relationship between optimal parameters $\delta_{0,u}$ and $\delta_{0,l}$ and the parent particle grade (in volume).

The parameters presented in Figure 3 are mainly associated with the tails of the curves shown in Figure 2, thus controlling the dispersion of the population of particles that can be found after breakage. When particles with lower grades are fragmented, the lower boundary tends to become more uniform and closely concentrated around lower grades. In contrast, the upper boundary exhibits higher dispersion, increasing the phase space in which progeny particles may be distributed after breakage. Figure 3 shows exponential decay trend for $\delta_{0,u}$ and an approximately exponential relationship for $\delta_{0,l}$, both in respect to the parent particle grade.

The relationships between the optimum values of $\delta_{u,1}$ and $\delta_{l,1}$ and the parent grade are presented in Figure 4. Parameter $\delta_{l,1}$ showed strong similarities to $\delta_{0,l}$, while $\delta_{u,1}$ remained nearly constant for parent grades below 45%, followed by an exponential decay at higher grades. These parameters control the curves associated with progeny sizes significantly larger than the liberation size (D_{lib}).

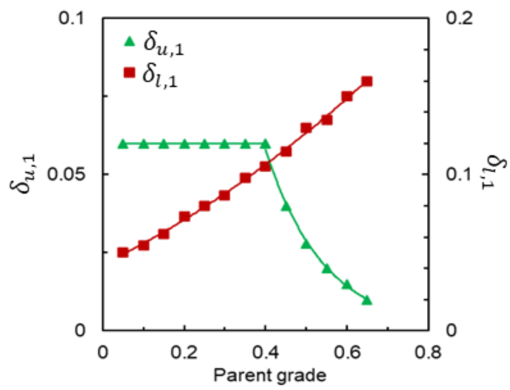

Figure 4. Relationship between optimal parameters $\delta_{u,1}$ and $\delta_{l,1}$ and the parent particle grade.

Finally, fitting of parameter λ (Eq. (6)) is presented in Figure 5 as a function of the parent grade. Results highlight a nearly parabolic relationship with maximum values for λ as 0.54, which was obtained for parent particles with 35% grade by volume. Increment in λ values correspond to a higher standard deviation of the inner distribution describing the population of middling particles. As such, results in Figure 5 demonstrate that particles with average composition are likely to produce a more dispersed population of particles in the different grade classes, while breakage of low- and high-grade parent particles tend to produce progenies with grades closer to the parent particle grade.

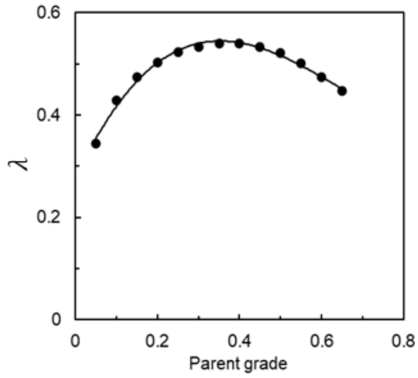

Figure 5. Relationship between optimal parameter λ the parent particle grade.

5. Conclusions

The present work uses a texture-dependent liberation model to calibrate a theoretical mathematical description of the Andrews-Mika liberation diagram.

The TLM simulates a conceptual ore texture and is used in virtual experiments. These experiments are performed varying the grade of the parent particle to map the liberation behaviour of the ore. Liberation spectra obtained from the TLM are then used as the basis to calibrate the boundaries and the internal distribution of the Andrews-Mika diagram. Results demonstrate a strong dependence of the model parameters with the grade of the parent particles, even though they were originally supposed to be constant [6,7], globally describing the ore texture.

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References

[1] Gaudin, A.M. (1939) Principles of Mineral Dressing. McGraw-Hill, New York, pp. 70–91.

[2] Andrews, J.R., Mika, T. (1975) Comminution of heterogeneous material: development of a model for liberation phenomena. Proc. 11th International Mineral Processing Congress, Cagliari, pp. 59–88.

[3] King, R.P., Schneider, C.L. (1998) Mineral liberation and the batch comminution equation. Minerals Engineering, 11(12), 1143–1160.

[4] Gay, S.L. (2004) Simple texture-based liberation modelling of ores. Minerals Engineering, 17, 1209–1216.

[5] Hilden, M.M., Powell, M.S. (2017) A geometrical texture model for multi-mineral liberation prediction. Minerals Engineering, 111, 25–35.

[6] King, R.P. (2001) Modeling and Simulation of Mineral Processing Systems. Butterworth-Heinemann, Oxford, pp. 45–80.

[7] Schneider, C.L. (1995) Measurement and Calculation of Liberation in Continuous Milling Circuits. PhD Thesis, University of Utah.