Triangular $C^2$ Interpolants by Piecewise Rational Functions

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In this paper a $C^2$ interpolation scheme on a triangle is presented. The interpolant assumes given values and derivatives of orders up to 2 at the vertices of the triangle. It is made up of partial interpolants blended with weight functions. Any partial interpolant with respect to a vertex is a piecewise quintic defined on a split of the triangle and interpolates the data at the vertex and on the two sides sharing the vertex, while the corresponding weight function is a rational polynomial in the barycentric coordinates. The resulted interpolant is a piecewise rational polynomials. By using the Bernstein-Bezier representation of polynomials, the interpolant is easy to describe and evaluate.

1. Introduction

Local interpolation over triangulations has received widespread discussion as a tool for use in free-form surface design in Computer Aided Geometric Design, finite element computation, and scattered data processing. The general setting for local interpolation of a prescribed smoothness is to setup a model on one single representative triangle. In order to attain global smoothness on the whole triangulation, a successful model must allow the interpolants on any two adjacent triangles satisfying the prescribed smoothness across the common edge. For the sake of computation, the interpolant has to be simple enough and could be a polynomial, a rational polynomial, and even their piecewise versions defined on a split of the model triangle. In practice, the $C^1$ and $C^2$ models are of essential importance. There have been many $C^1$ schemes but few for $C^2$ ones; see Farin (1986), Farin (1990), Goodman and Said(1991), Peters (1990), and Strang and Fix (1973). Since $C^1$ schemes supersede $C^0$ ones in better approximation and visual effect (esthetic feeling), endeavor has been paid in designing $C^2$ schemes (Alfred, 1984; Wang, 1992).

In general, a $C^r$ scheme of polynomial requires vertex data of order $2r$, and the degree of the polynomial can reach $4r+1$. So for a $C^2$ scheme, the derivatives assigned at vertices must be of orders up to 4 and the degree of the polynomial has to be 9 (Zenisek, 1970). Higher degrees mean more complexity and instability. Ideas to overcome this are to use rational polynomials or to split the macrotriangle into microtriangles on which a $C^2$ spline is to be defined; see Alfred (1984) and Zhan (1996) for instance.

The drawback lying in the splitting trick is that the splitting causes more microtriangles. For this, a rational polynomial interpolant can be adopted. Alfeld and Barnhill (1984) developed a transfinite $C^2$ scheme the discrete version of which results in rational interpolant. Liu and Zhu (1995) characterized $C^2$ rational schemes and presented certain $C^2$ discrete triangular interpolants. Similar schemes can be found in Zenisek (1970).

The rational approach is able to prevent the triangle $T$ from being split. Nonetheless, this approach exhibits a possible flaw that the degrees of the denominator and numerator could still be very high. For instance, for the scheme with quintic precision in Whelan (1986), the rational interpolant has degrees (9, 4) of the pairs of the denominator and numerator.

By combining the rational formation with the splitting technique, in the present paper, we’ll develop a $C^2$ scheme in a quite natural way. The interpolant is a weighted sum of three partial interpolants, each of which is a $C^2$ piecewise quintic on a split of the triangle $T$ with respect to a vertex. And the weight functions are rational polynomials of degrees $(2, 2)$. Then the degrees of the resulted interpolant are $(7, 2)$.

The paper is organized as follows. In §2, as a preliminary, we formulate bivariate quintics in Bernstein-Bezier forms. Then in §3, the $C^2$ triangular interpolation scheme is described and the explicit formulation of the interpolant is displayed in §4. At last, a conclusion is made in §5.
2. A formulation of quintic polynomials

Take $T = A_0A_1A_2$ as a triangle. Let $e_i = A_{i+1}A_{i-1}$ denote the opposite edge of $A_i$ and $n_i$ the outer normal vector of $e_i$, $i = 0, 1, 2$, counting modulo 3. For $a = (p, q)$ with $|a| = p + q$, denote by $D^a$ the partial differential operator $\partial^{p+q}/\partial x^p \partial y^q$. Suppose $f \in C^2(T)$, for a $C^2$ scheme, an interpolant $g \in C^2(T)$ of $f$ must at least satisfies $D^a (g - f)(A) = 0$, $0 \leq |a| \leq 2$, $i = 0, 1, 2$. Let $A_0 \in R^2$, $A_1 \in R^2$ form a triangle $T = A_0A_1A_2 \subset R^2$. For $A = (x, y) \in R^2$, denote by $u = (u_0, u_1, u_2)$ its barycentric coordinates (see Farin (1986) for the detail) with respect to $A_0, A_1$, and $A_2$ (or to $T$), i.e.

$A = u_0A_0 + u_1A_1 + u_2A_2$, with $u_0 + u_1 + u_2 = 1$.

An index $\lambda = (\lambda_0, \lambda_1, \lambda_2)$, $\lambda_i \in Z_n$, has length $|\lambda| = \lambda_0 + \lambda_1 + \lambda_2$. A polynomial $p \in P_6$, the set of all the bivariate polynomials of degrees up to $k$, can be expressed in Bernstein-Bezier form (short for $B$-form)

$p(u) := p(x, y) = \sum_{i = 0} b_i \phi_i(u), \quad B_{\lambda, A}(u) = \frac{kl}{\lambda_1 \lambda_2 !} u_0^{\lambda_0} u_1^{\lambda_1} u_2^{\lambda_2}$

where $b_i$ is the $B$-ordinates at index $\lambda$ with the Bernstein basis function $B_i(u)$.

Let $e \in R^2$ be a vector $e \in \theta_0 A_0 + \theta_1 A_1 + \theta_2 A_2$, with $\theta_0 + \theta_1 + \theta_2 = 0$, and $\theta = (\theta_0, \theta_1, \theta_2)$. Take $\varepsilon_\lambda = (1, 0, 0)$, $\varepsilon_1 = (0, 1, 0)$, and $\varepsilon_2 = (0, 0, 1)$. Define recursively

$b^{(0)} = b_\lambda, \quad |\lambda| = k; \quad b^{(r)} = \frac{kl}{(k-r)!} \sum \lambda r! \lambda e | \lambda| = k - r.$

Then the $r$-th derivative of $p$ along direction $e$ is formulated as

$\partial^r/e p(u) = \frac{kl}{(k-r)!} \sum b^{(r)}(\theta) B_{k-r, A}(u).$

Take $e_i = A_{i+1}A_{i-1}$ as the vector from $A_{i+1}$ to $A_{i-1}$ opposite to $A_i$, $i \in Z_3$. Then $n_i = e_i - h_i e_i$ with $h_i = (e_i \cdot e_i)/(e_i \cdot e_i)$ is an outer normal vector to $e_i$, $i \in Z_3$. Let $V_i = (A_{i+1} + A_{i-1})/2, V_i = (A_{i+1} + A_{i-1})/3, V_i = (A_{i+1} + A_{i-1})/3$.

Taking $A_\Lambda = \{ \lambda = (\lambda_0, \lambda_1, \lambda_2); |\lambda| = 5 \}, A_\Lambda = \{ \lambda \in A_\Lambda; \lambda_2 \geq 3 \}, A_\Lambda = \bigcup_{\lambda \in A_\Lambda} A_\lambda$ and $A_\Lambda = \{(1, 2, 2), (2, 2, 1), (2, 2, 2) \}$, and $E_\lambda = p(A_\lambda), F_\lambda = \partial_\lambda X A(A_\lambda), G_{\lambda k} F_\lambda = \partial_{\lambda k} X A A(A_\lambda), i, j, k \in Z_3, j, k \neq i$.

we have (Farin, 1990)

**Lemma 2.1** Let $p$ be a quintic of the form $p(u) = \sum_{i,j,k} b_{i,j,k} A_{i,j,k}(u)$. Define $q(u) = \sum_{i,j,k} b_{i,j,k} A_{i,j,k}(u)$. Then

$q(u) = \sum_{i,j,k} E_{i,j,k} \phi_i(u) + \sum_{i,j,k} F_{i,j,k} \phi_j(u) + \sum_{i,j,k} G_{i,j,k} \phi_k(u)$

where

$\Phi_i(u) = u_i^3 (10 - 15 u_j + 6 u_k^2), \quad \Phi_j(u) = u_i^3 u_j (4 - 3 u_i), \quad i \neq j,$

$\Phi_k(u) = u_i^3 u_k i, j, k \neq 0, 1, 2.$

This lemma implies that the B-coordinates of $p$ with indices $A_\Lambda$ are determined by the data assigned at the vertices of $T$. Further, given 3 normal derivatives on one side of $T$, the $B$-ordinates with $A_\Lambda$ can be determined.

**Lemma 2.2** On the basis of Lemma 2.1, $b_{122}, b_{212}$ and $b_{221}$ are determined in addition by

$\partial_{\lambda_0} X A p(V_{0}) \phi_i(u), \quad \partial_{\lambda_0} X A p(V_{0}) \phi_i(u), \quad$ and $\partial_{\lambda_0} X A p(V_{0}) \phi_i(u),$

$b_{122} = -\frac{1}{12} \partial_{\lambda_0} X A p(V_0) + \frac{1}{6} \left[-(b_{114} + 4 b_{113} + 4 b_{112} + b_{110}) (b_{215} + 4 b_{214} + 6 b_{213} + 6 b_{212} + 6 b_{211})

- h_0 (b_{114} + 4 b_{113} + 4 b_{112} + 4 b_{111} + b_{110}) \right], H,$

$b_{221} = \frac{3}{40} [2 \partial_{\lambda_0} X A p(V_0) - \partial_{\lambda_0} X A p(V_0)] - \frac{5}{6} b_{110} + \frac{1}{3} b_{111} - b_{112}. \quad b_{212} = \frac{3}{40} [2 \partial_{\lambda_0} X A p(V_0) - \partial_{\lambda_0} X A p(V_0)] - \frac{5}{6} b_{110} + \frac{1}{3} b_{111} - b_{112}. \quad$
Lemma 2.3 With the assumptions that $\partial_{\nu_0}p$ and $\partial_{\nu_0}^2 p$ on edge $e_0$ reduce to polynomials of degrees 3 and 1, respectively, the formulas in Lemma 2.2 are replaced with

$$b_{22} = \frac{1}{6}([b_{00} + 4b_{31} + 4b_{13} - b_{04}] + (1 + h_1)(b_{10} + 4b_{41} + 6b_{32} - 4b_{13} + b_{34}) - h_1(b_{11} + 4b_{32} + 6b_{33} - 4b_{13} + b_{34})).$$

$$b_{21} = (2b_{00} + b_{03})/3 - b_{02}, \quad b_{12} = (2b_{00} + b_{03})/3 - b_{02}.$$  

In the sequel, we’ll use $b^i_j, \lambda \in \Lambda_c$, to specify the $B$-ordinates $b^i_j$ obtained with respect to the data on the edge $e_i, i \in Z_3$.

3. The $C^2$ interpolation scheme

In this section, we’re presenting a $C^2$ interpolation scheme. Given a triangulation $T$ of a polygonal region $\Omega \subset R^2$, and integers $k$ and $r$, $k \geq r \geq 0$, the spline space of piecewise $C^r$ polynomials of degree $k$ is defined as (Wang, 2001)

$$S^k_\tau(T) = \{s \in C^r(\Omega); s|_T \in P_{k+1}, \sigma \text{ being any triangle in } T\}.$$  

3.1 Partial interpolants

For the triangle $T = A_0A_1A_2$, take

$$B_{m} = (1-t_m)A_{11} + t_mA_{12}, \quad 0 < t_m < t_m < 1, \quad i \in Z_3, \quad m = 1, 2.$$  

For any $i \in Z_3$, by connecting $A_i$ and $B_{m}, m = 1, 2$, we form a split $\Delta_i$ of $T$ with subtriangles $T_1 = A_0A_1B_m, T_2 = A_0B_1B_2,$ and $T_3 = A_1B_2B_3.$ See Figure 1. By using cofactor conforming approach developed by Wang(2001), it’s easy to verify that $\dim S^2_\tau(\Delta_i) = 33.$

![Figure 1: The partial split $\Delta_0$](image1.png)  

![Figure 2: The stencil of $[b^i_j ; l, l = 1, 2, 3]$](image2.png)

Now consider the following partial interpolation problem for $i \in Z_3$.

**Interpolation Problem (P)** For $f \in C^2(T)$, find $s_i \in S^2_\tau(\Delta_i)$ such that

$$D^\sigma(s_i - f)(A_i) = 0, \quad 0 \leq \sigma \leq 2, \quad j \in Z_3;$$  

$$\partial^\nu(s_i - f)(V_i) = 0, \quad \partial^\nu_\nu(s_i - f)(V_{m}) = 0, \quad l \neq i, \quad l \in Z_3, \quad m = 1, 2, 3.$$  

Suppose $s_i$ is an interpolant of (P), denote $s_i = s_i|_{T_0}$ for $i \in Z_3$ and $i = 1, 2, 3$.

Note that the number of conditions given in problem (P) is 27, which is 6 less than the dimension of the underlying interpolation space $S^2_\tau(\Delta_i)$.

For the sake of simplicity, without loss of generality, the discussion in this subsection is only on $\Delta_0.$ Suppose $s_0$ is an interpolant of (P) and

$$s_0 = \sum_{|l|=5} b_{l1}B_{l1}(x_i)$$  

where $(x_i, y_i) \in R^2$ with respect to $T_0.$

Note that for any $A = (x, y) \in R^2$.

$$A = u_0A_0 + u_1A_1 + u_2A_2 = u_{11}A_0 + u_{13}A_1 + u_{23}B_1 = u_{02}A_0 + u_{22}B_2 + u_{13}A_1 + u_{23}B_2,$$

and then $u_0 = u_{01} = u_{02} = u_{03}.$ Base on this observation, each $s_i$ can be written as
\[ s_0(u) = \sum_{k=0}^{5} \frac{u_{k+1} - u_k}{5} g_k(u_{i+1}, u_{i+2}) \]

where \( g_k \) is a univariate polynomial of degree \( k \), which can be seen as the \( k \)-th layer of \( s_0 \). This allows \( s_0 \) another formulation

\[ s_0(u) = \sum_{k=0}^{5} \frac{u_{k+1} - u_k}{5} g_k(u_{i+1}, u_{i+2}) \]

where \( g_k \) is a univariate piecewise polynomials of degree \( k \), \( k = 0, 1, \ldots, 5 \), with successive pieces \( g_{k_1}, g_{k_2} \), and \( g_{k_3} \), defined on the partition \( 0 < t_1 < t_2 < 1 \).

The supports of the \( B \)-ordinates are depicted in Figure 2. From Lemma 2.1, the \( B \)-ordinates marked with \( \bullet \) are determined by conditions (1), and from Lemma 2.2, the conditions (2) in addition determine those marked with * in Figure 2.

In particular, the \( g_k \) with \( k = 0, 1, \) and \( 2 \), are all overall polynomials determined by conditions (2) and it is easy to see that, with \( C^2 \) relations, \( g_2 \) is uniquely solved by the previously determined \( B \)-ordinates. Refer to Schumaker (1981) for univariate splines.

For \( g_2 \) and \( g_5 \), we make the following convention that

(i) \( g_5 \) reduces to an overall quintic;

(ii) \( g_2 \) is under \( C^2 \) relations and some two successive pieces of \( g_2 \) reduce to a quartic.

Under restriction (i), \( g_5 \) is obviously determined by conditions (1). The rule (ii) means that \( g_2 \) is a \( C^2 \) cubs of two pieces with 6 parameters, just the same as the number of * and \( \bullet \) on \( g_4 \). Then we conclude that

**Theorem 3.1** Under the restrictions (i) and (ii), there uniquely exists an interpolant \( s \in S^2_5(\Delta_3) \) satisfying the conditions (1) and (2).

### 3.2 The macro interpolants

Similar to \( s_0 \), the partial interpolants \( s_1 \in S^2_3(\Delta_4) \) and \( s_2 \in S^2_3(\Delta_4) \) of \( (P) \) are obtained uniquely under (i) and (ii).

From the discussion above, we can also see that the partial interpolants \( s_i, i \in \mathbb{Z}_3 \), hold

\[ s_i | e_1 = s_{i+1} | e_2 = s_{i+2} | e_i, \quad c_{i+1}^i | s_{i+1} | e_i = c_{i+2}^i | s_{i+2} | e_i, \quad r = 1, 2. \]

Define weight functions \( w_i(u) = u_1^2 / (u_0^2 + u_1^2 + u_2^2) \), \( i \in \mathbb{Z}_3 \). Obviously, these functions have the properties that \( w_i \in C^\infty \), \( \sum_{i \in \mathbb{Z}_3} w_i(u) = 1, w_i(u) |_{e_i} = 0 \), and \( c_{i+1}^i w_i(u) |_{e_i} = 0, i \in \mathbb{Z}_3 \).

![Figure 3: The stencil of \( s_0 \) in the uniform BCS u.](image)

Now it’s ready to the main result of this paper.

**Theorem 3.2** Suppose \( s_i \in S^2_3(\Delta_4), \quad i \in \mathbb{Z}_3 \), are interpolants of \( (P) \) for \( f \in C^2(\Delta) \) under the restrictions of (i) and (ii). Then the blending function

\[ f_r(u) = \sum_{i \in \mathbb{Z}_3} w_i(u) s_i(u) \]  

satisfies that

\[ D^\alpha(f_r - f)(A) = 0, \quad 0 \leq |\alpha| \leq 2, \quad i \in \mathbb{Z}_3 \]

\[ c_{i+1}^i (f_r - f)(V_i) = 0, \quad c_{i+2}^i (f_r - f)(V_m) = 0, \quad i \in \mathbb{Z}_3, \quad m = 1, 2. \]

Still, \( c_{i+1}^i f_r |_{e_i}, \quad i \in \mathbb{Z}_3 \), are polynomials of degrees \( 5 - r \), \( r = 0, 1, 2 \). In addition, \( f_r \) reproduces polynomials of degree 5.
Theorem 3.3 Suppose \( s_i \in S_h^b(\Delta_i), i \in Z_3 \), satisfies \( D^\alpha(s_i - f)(\Delta_i) = 0, 0 \leq \alpha \leq 2, j \in Z_3 \); \( \hat{c}_{\alpha} s_i \mid e_i \) and \( \hat{c}_{\alpha}^2 s_i \mid e_i \) are of degrees 3 and 1, respectively. Then \( f \) is defined as in (4). Then \( f \) reproduces polynomials of degree 3 and satisfies (5). On \( e_i \), \( \hat{c}_{\alpha} f \) and \( \hat{c}_{\alpha}^2 f \), \( i \in Z_3 \), are polynomials of degrees 3 and 1, respectively.

The usage of rational functions can be dated back to the well-known BBG scheme (Barnhill et al., 1973) and Nielson’s side-vertex scheme (Nielson, 1979). The interpolation schemes in Liu and Zhu (1995) and Xu et al. (2000) can also be regarded as side-vertex schemes, since a partial interpolant on a triangle in those schemes interpolates data at a vertex and on its opposite side. In our method, a partial interpolant interpolates data on two sides sharing a vertex and then our interpolation can be called a side-side scheme. The weight functions used in side-vertex schemes and in side-side schemes are different. For instance, the weight functions used in Liu and Zhu (1995) are

\[
\omega_i(u) = u_{i1}^2 u_{i1}^2 f(u_{i1}^2 + u_{i2}^2 + u_{i3}^2). \quad i \in Z_3.
\]

Obviously, the macro interpolant as a whole is a piecewise rational function with degrees \((7, 2)\) for the pairs of the denominator and numerator. By contrast, for the scheme with quintic precision in Liu and Zhu (1995), the rational interpolant has degrees \((9, 4)\).

4. Evaluation of the interpolant

Suppose \( f \in C^3(\mathcal{T}) \). Let \( q(u) = \sum b_i B_{ui}(u) \) be the polynomial in Lemma 2.1, with the symbol \( p \) replaced with \( f \). Then \( q \) is nothing but the polynomial interpolating function \( f \) at the vertices of \( \mathcal{T} \) such that

\[
D^\alpha(q - f)(\Delta_i) = 0, \quad 0 \leq \alpha \leq 2, \quad i \in Z_3.
\]

Now fix one \( i \in Z_3 \). By assigning \( \hat{c}_{\alpha}(p - f)(V_i) = 0, \hat{c}_{\alpha}^2(p - f)(V_m) = 0, m = 1, 2 \), the \( B \)-ordinates \( b_i, \lambda \in \Lambda \), can be determined via Lemma 2.2.

Continuing the preceding, we will give an expression of the partial interpolant \( s_i \in S_h^b(\Delta_i), i \in Z_3 \), and hence the macro interpolant \( f \). In order to express \( s_i \), owing to the symmetry, we only illustrate the formulation of \( s_0 \), and the others can be obtained by rotation with the index \( i \).

Note that a \( B \)-form polynomial in a barycentric coordinates system (BCS) can be expressed in any other BCS with an affine transform between the two systems. For the interpolant \( s_i \in S_h^b(\Delta_i) \), we prefer to write

\[
s_i(u) = s_0 | T_0 = \sum b_i B_{ui}(u) = \sum_{i,j} b_i B_{ui}(u) = \sum_{i,j} b_i B_{ui}(u_{j1} u_{j2}), \quad i = 1, 2, 3
\]

in the uniform BCS \( u \) with \( T \) rather than that with \( T_0 \) as in (3). This form has the advantages that

\[
b_{22:1} = b_{21:2}, \quad b_{22:1} = b_{22:1} = b_{22:1}, \quad b_{22:1} = b_{22:1}, \quad b_{21:2} = b_{22:1}, \quad b_{22:1} = b_{22:1}, \quad b_{21:2} = b_{22:1}.
\]

For \( \Lambda_1 = 1, 2 \), \( b_{12} = b_{12} \), for \( \Lambda_2 = 1, 2 \), \( b_{11} = b_{11} \), for \( \Lambda \in \Lambda_1 \), \( b_{13} = b_{13} \), for \( \Lambda \in \Lambda_2 \).

Since the segments \( A_0 B_0 \) and \( A_2 B_2 \) are on the lines

\[
T_0^\gamma : \quad \gamma_1 = t_0 u_1 - (1 - t_0) u_2, \quad T_0^\gamma : \quad \gamma_1 = t_0 u_1 - (1 - t_0) u_2,
\]

respectively, we can write

\[
s_i(u) = \sum b_i B_{ui}(u_{j1} u_{j2}) + 5ug_3(u_{j1}, u_{j2})
\]

where \( g_3 \) and \( g_4 \) are piecewise \( C^2 \) cubics and quartics in \( u_1 \) and \( u_2 \), and

\[
g_3 = \begin{cases} g_{31}, & \gamma_1 \geq 0, \\ g_{32}, & \gamma_1 < 0 < \gamma_2, \\ g_{33}, & \gamma_2 \leq 0, \end{cases} \quad g_4 = \begin{cases} g_{41}, & \gamma_1 \geq 0, \\ g_{42}, & \gamma_1 < 0 < \gamma_2, \\ g_{43}, & \gamma_2 \leq 0, \end{cases}
\]

4.1 The representation of \( g_3 \)

The unknown \( B \)-ordinates for \( g_3 \) include \( b_{201:1}, b_{201:2}, b_{221:1}, b_{221:2}, b_{221:3}, b_{221:3}, b_{203:2}, \) and \( b_{23:3} \). We only provide the formula of \( b_{203:1} \) and the other can be derived similarly.
The \( C^2 \) smoothness of \( g_t \) implies that there are two numbers \( c_1 \) and \( c_2 \), such that

\[
g_{t1}(u_t, u_z) + c_1[v_2(u_t, u_z)]^3 = g_{x2}(u_t, u_z), \quad g_{x2}(u_t, u_z) + c_2[v_2(u_t, u_z)]^3 = g_{t2}(u_t, u_z).
\]

Identifying the coefficients of \( u_t \) and \( u_z \) in these equations, we get

\[
c_1 = \frac{(b_{21,3} - b_{21,2})(t_{02} - 1) - (b_{21,2} - b_{21,1})t_{02}}{t_{01}(t_{01} - t_{02})}, \quad c_2 = \frac{(b_{22,3} - b_{22,2})(t_{02} - 1) - (b_{22,2} - b_{22,1})t_{02}}{t_{01}(t_{01} - t_{02})}.
\]

\[
b_{003,3} = b_{003,3} - c_0(t_{01} - 1)^3 - c_2(t_{02} - 1)^3.
\]

**4.2 The representation of \( g_t \)**

Under (ii), \( g_t \) is of \( C^2 \), and two pieces of \( g_t \), e.g. \( g_{t1} \) and \( g_{t2} \) are identical, i.e.

\[
g_{t1}(u_t, u_z) = g_{x2}(u_t, u_z), \quad g_{x2}(u_t, u_z) + d_2[v_2(u_t, u_z)]^3 = g_{t2}(u_t, u_z),
\]

where \( d_2 \) is the \( C^2 \) cofactor of \( g_t \) on \( G_2 \). Then

\[
d_2 = \frac{b_{22,3} - b_{22,1}}{t_{02}(1 - t_{02})}, \quad b_{13,1} = b_{13,3} + d_2 t_{02}(1 - t_{02})^3, \quad b_{04,1} = b_{04,3} + d_2(1 - t_{02})^3.
\]

The other \( B \)-ordinates \( b_{140,2}, b_{131,2}, b_{222,2}, b_{113,2}, b_{042,2}, b_{403,3}, \) and \( b_{131,3} \) are formulated similarly and omitted here.

**5. Conclusion**

We have shown in this paper a \( C^2 \) local interpolation method and the detailed representation of the interpolant. In literature, Alfeld (1984) developed a \( C^2 \) scheme on the so-called twice HCT split for only \( C^2 \) data. Whelan (1986) displayed a \( C^2 \) interpolant of piecewise nonics with \( C^2 \) data, which indeed a special case of the schemes in Zenisek (1970). Both are comparatively complicated because of the large number of split pieces of the macrotriangle of the former and the high degree of polynomials and high order of derivatives of the latter. We can see that the scheme in this paper overcomes these drawbacks to a large extent. With the explicit formulation of the interpolant with Bernstein-Bezier technique, it’s also easy to implement the evaluation of this interpolation scheme.

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