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An Investigation on the Stability of a Turbidostat

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A turbidostat is a continuous biochemical reactor in which the biomass concentration is controlled by the manipulation of the dilution rate. In this paper, some issues concerning the stability and the basins of attraction for the steady state point are discussed for a preliminary simple case. The biomass growth is supposed to follow the Monod kinetics and the controller acts according to a PI logic.

1. Introduction

A turbidostat is a fixed volume bioreactor in which the biomass concentration is kept constant by manipulating the inlet flow rate inside a feedback control system. In its first application (Bryson et al., 1953) the regulation mechanism was quite simple. An optical device for the measurement of the turbidity allowed to quantify the biomass concentration and control it through a relay device which opened the feeding valve at a fixed extent each time the set point value was overtaken.

The peculiarity of turbidostat is that it can operate at a fixed growth velocity chosen by the operator; furthermore its dynamics shows some interesting behaviours as proved by the experimental observations performed by Davey et al. (1998) who described the existence of chaotic regimes for this system with an aerobic yeast cultivation.

In this paper, the biomass concentration is supposed to be regulated by a PI controller which acts on the inlet flow rate by changing the dilution rate of the system. The role of the controller will be evaluated for its influence on both the stability of steady state regimes and the amplitude and the structure of the basins of attraction.

2. The mathematical model

It is assumed that an unstructured and un-segregated model is accurate enough to describe the growth kinetics.

The dimensionless equations that describe the physical system are reported in Table 1. The dimensionless state variables are the biomass concentration, $x(t)$, the substrate concentration, $s(t)$, and the value of the dilution rate computed by the controller, $dil(t)$. $\mu(s)$ indicates the growth kinetics, $y(s)$ is the biomass yield for the substrate, whereas K_a, θ_I and x_{sp} are respectively the dimensionless proportional gain, the integral time and the set point value for the controller. The microbial death rate is supposed to be negligible.

In principle, the controller manages the biomass level by increasing the expulsion rate of the micro-organisms from the reactor when they are in excess, and by reducing it, in the opposite case. It is important to notice that, even if the dilution rate is an inherently nonnegative quantity, the controller can also indicate a negative value for the manipulated variable, especially when the biomass level is very low. This can be classified as a windup phenomenon and the final control element keeps the saturation value (*dil*=0) until the level of biomass becomes high enough to determine a change in the sign of the computed dilution rate. From the dynamical standpoint, this makes the system a hybrid system: in fact, is described by a C^0 piecewise function, in which a shift condition (*dil*=0) describes the transition between two smooth vector fields (Leine, 2006). In this case, the system obtained for dil≤0 is a batch reactor in which the controller output is still calculated according to the PI logic: in this half-space of the state space, the biomass level increases to the expenses of the residual substrate content until its amount is high enough to make the calculated dilution rate positive again. Obviously, it may also happen that the biomass level and the substrate amount are not sufficient to determine the transition in the upper half-space and, in this case, the reactor will extinguish (it is like a death phase for a batch reactor). In this case it is possible to prove that the system diverges (i.e., *dil*(*t*) approaches asymptotically -∞).

Table 1. Equations of the mathematical model.

If $dil(t) > 0$		*If* $dil(t) \le 0$	
$x'(t) = [\mu(s(t)) - dil(t)]x(t)$	(1)	$x'(t) = \mu(s(t))x(t)$	(4)
$s'(t) = dil(t)(1 - s(t)) - \dfrac{\mu(s(t))}{y(s(t))}x(t)$	(2)	$s'(t) = -\dfrac{\mu(s(t))}{y(s(t))}x(t)$	(5)
$dil'(t) = K_a\left[x'(t) + \dfrac{1}{\theta_I}(x(t) - x_{sp})\right]$	(3)	$dil'(t) = K_a\left[x'(t) + \dfrac{1}{\theta_I}(x(t) - x_{sp})\right]$	(6)

In this paper, the kinetic expression that is adopted is a simple Monod growth with a constant biomass yield.

$$\mu(s) = \frac{s}{k + s} \qquad (7) \qquad\qquad y(s) = 1 \qquad (8)$$

3. Steady states

The determination of the steady states and of their stability is quite straightforward. In the lower upper half-space (*dil*≤0) there is an infinite family of steady states parameterised by the variable $\alpha \in \mathbf{R}^-_0$ given by $Q_\alpha = (x_{sp}, 0, \alpha)$ which can be proved to be all unstable (the coordinates are respectively the biomass concentration, the substrate concentration and the dilution rate). The one with $\alpha=0$ lies on the switch surface. In the upper half-space (*dil*>0), it is possible to show that for $x_{sp} \geq 1$ there are no steady states (such values are physically meaningless because $x_{sp}=1$ corresponds to the complete conversion of the substrate). For $x_{sp}<1$ there is a single steady state given by

$$P_{ss1}(x_{ss1}, s_{ss1}, dil_{ss1}) = (x_{sp}, 1-x_{sp}, (1-x_{sp})/(k+1-x_{sp})) \qquad (9)$$

which is the only nontrivial operating condition for the reactor. Obviously, the set point value is the only controller parameter which influences the occurrence of steady states. In order to describe the stability of the first steady state point, the characteristic polynomial of the Jacobian matrix evaluated at P_{ss1} is computed. The decomposition in a first order factor with a negative zero and in a second order polynomial with no sign variations indicates the stability of the steady state points independently on the controller gain and integral time values.

$$p(\lambda)=\left(\frac{1-x_{sp}}{k+1-x_{sp}}+\lambda\right)\left(\frac{K_a x_{sp}}{\vartheta_I}+\left(K_a+\frac{k}{(k+1-x_{sp})^2}\right)\lambda+\lambda^2\right) \tag{10}$$

4. Basin of attraction

Though the controller parameters do not influence the position and the stability of steady states for a population described by a Monod growth, their effect is quite evident in transient behaviours: in fact the choice of these quantities influences both the quickness of response and basins of attraction of a stable steady state.

The importance of the determination of basins of attraction is connected essentially with two applications: the first one is the choice of a set of initial conditions that can lead the system to approach the optimal operating regime; the second one, which is particularly important in control issues and is the objective of this paper, is the determination of the significance of perturbations that the system is able to withstand without leaving toward an undesired asymptotic regime.

The investigation about the basins of attraction of a stable regime can be a complex task. It consists in the determination of their boundary which is the stable manifold of an unstable invariant set for the system (Ott, 2002) and whose structure can also be very complicated, e.g., fractal (Grabogi et al., 1983). In this case, the almost complete absence of information about the nature of the basins of attractions for hybrid systems in literature convinced us to lead the computation with a brute force simulation approach. It is worth noting that the closest unstable steady state to the stable one lies on the switch surface and, hence, it does not exists a neighbourhood of such point in which the vector field is C^1 regular: this prevent us from computing the stable manifold using the stable manifold theorem on this point because its existence is not guaranteed at all.

In order to simplify the computation, it is supposed that the perturbation from the steady state value involves only the state variables *x* and *s*, but not *dil*. This assumption can be justified observing that *dil* is not the actual value of the dilution rate but the quantity computed by the controller and, hence, not subject to disturbances. This allows to simplify the proposed task because the objective is not to determine the whole basin of attraction in the three dimensional state space, but only its intersection with the plane given by $dil=dil_{ss1}$.

The simulations were performed using the built-in Mathematica 5® solver for ODE integrated with a shifting system which allows switch between the two vector fields each time the transition surface is crossed.

4.1 Effect of the controller parameters on the basin of attraction

In order to determine the effect of the controller parameters K_a and θ_I on the basin of attraction, a group of simulations are performed adopting the following parameter values: k=1 x_{sp}=0.9

A high value has been chosen for the set point value (that must always be less than 1) in order to verify the response of the system which must approach a steady state (given by the point P_{ss} (0.9, 0.1, 9.09·10^{-2}) close enough to the shift surface.

A first set of simulations shows the appearance of the section of the basin for a constant K_a (0.1) and different values of θ_I (0.1, 1.0 and 10). The results are reported in Fig. 1.

In Fig. 2 the effect of the proportional gain is evaluated at three different values (0.1, 1.0 and 10) for a constant integral time (10)

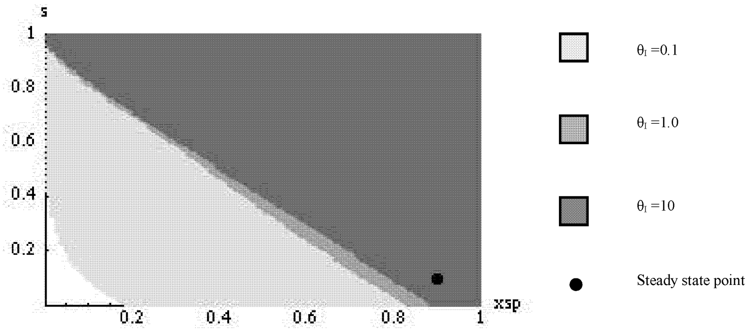

Figure 1 Intersection of the basin of attraction of the stationary point P_{ss} with the plane dil=9.09 10^{-2} for K_a=0.1 and for different values of θ_I.

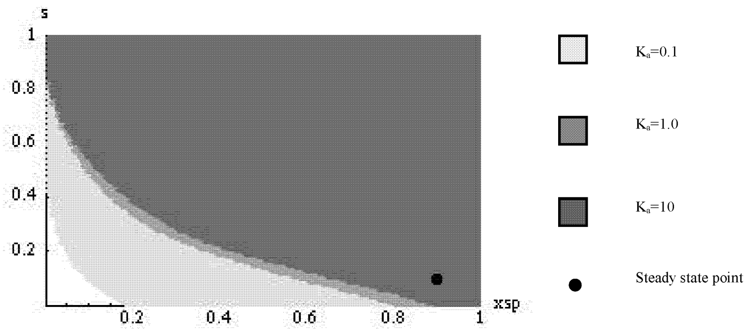

Figure 2 Intersection of the basin of attraction of the stationary point P_{ss} with the plane dil=9.09 10^{-2} for θ_I. =10 and for different values of K_a.

A first obvious consideration must be done about the effectiveness of the controller. Indeed, perturbations that increase the substrate concentration or the biomass content are well tolerated by the system, which manages to recover for almost every choice of the controller parameters. Conversely, a diminution of these two state variables, especially if simultaneous, may cause the extinction of the reactor itself. When the controller parameters are strict (high proportional gains and low integral times), the basin of attraction tends to shrink. As it can be observed for the selected values in fig. 1, the intersection of the basin with the dil=dil_{ss} plane is entirely included in the intersection obtained for a higher value of the integral time for a fixed K_a. The same argument can be repeated for decreasing proportional gain at a fixed integral time (Fig. 2). It is worth noting that the strictest choices of parameters determine a dangerous approach of the steady state point to the boundary of the basin, on the contrary, the loosest values allow a satisfying margin of stability for the system.

4.2 Transitions across the switch surface

In this paragraph, some elements about the time evolution of a perturbed system are discussed: in particular the occurrence of windup phenomena during the approach toward the stable stationary state will be analysed. In Fig. 3, the intersection of the basin of attraction with the $dil=dil_{ss}$ for K_a=1 and θ_I =0.1 is decomposed in three different

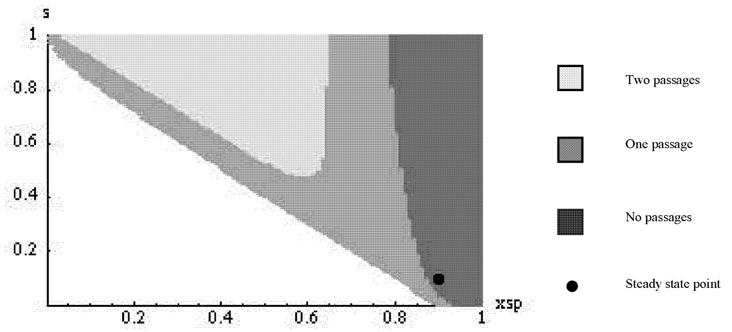

Figure 3. Decomposition of the intersection of the basin of attraction of the stationary point P_{ss} with the plane dil=9.09 10^{-2} for K_a=1.0 and θ_I =0.1 in different sets: the grey tone indicates how many passages in the lower half-spaces are necessary to recover to the steady state level.

domains: the darkest grey one is the set of perturbed conditions which manage to reach the stable steady state without windup occurrence; the medium grey set includes the perturbation points whose time evolution shows two crosses through the switch surface before approaching the steady state and the light gray one is made of points which requires four crosses. In this figure it is possible to notice that when the perturbation maintains a biomass level close to the set point value, a simple regulation of the valve can establish the stationary regime again; on the contrary, when the perturbation effect causes a significant decrease in biomass concentration, the system is compelled to shift to a batch operating condition for a certain period of time in order to reach again the concentration prescribed by the controller. In particular, there is a zone, characterised by high concentration of substrate and low level of biomass in which this recovery requires two passages.

In Fig. 4, the time evolution of the reactor from an initial condition belonging to this last category, i.e., (x(0),s(0))=(0.5, 0.7). The dilution rate is promptly reduced by the controller up to zero in order to avoid further elimination of biomass and, then, the reactor stays for about 4 dimensionless time units in a batch state consuming a consistent part of the residual substrate and increasing significantly the biomass level.

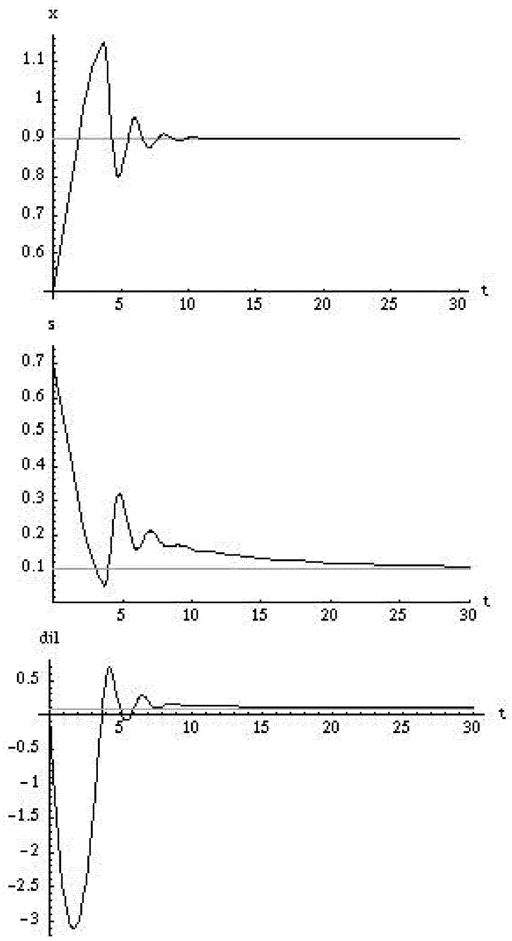

Figure 4: Time evolution of the state variables from the perturbed condition $x(0)=0.5$; $s(0)=0.7$

It is interesting to note in Fig. 3 how the boundary of the basin of attraction is in contact with points which require only two crosses through the switch surface: this line represents the border between the initial conditions that require a passage in the batch condition in order to allow the system to recover the biomass concentration prescribed by the reactor and the points from which the system shifts to the batch condition but does never manage to get out of it. An higher number of passages in the batch condition is less dangerous than a single one because it indicates that the controller produces an underdamped oscillatory response which is leading the system to the stable steady state (and, hence, the permanence in the batch condition is shorter and shorter).

5. Conclusions

A microbial population cultivated in a turbidostat according to a simple growth-associated Monod kinetics is able to maintain constant growth conditions even for operating conditions which are difficult to manage. A choice of quite loose controller parameters is a good compromise between the quickness of the response and the stability of the system to state perturbations. Passages through a batch working condition are a recovery procedure for the bioreactor when the biomass level becomes much lower than the set point value.

6. References

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