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COARSE – GRAINED BIFURCATIONS AND SYMMETRY BREAKING OF MAJORITY RULE DYNAMICS EVOLVINGS ON COMPLEX NETWORKS

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Our analysis is focused on the celebrated majority-rule model evolving on complex graphs. We prove analytically the symmetric properties of the network model and derive the conditions with respect to the network topology under which the symmetry breaks. We also address how the Equation-Free approach can be exploited to bridge in a computational rigorous way the micro and macro scales of the dynamics of stochastic individualistic neuronal models evolving on complex random graphs. In particular, we show how bifurcation analysis can be performed bypassing the need to extract macroscopic models in a closed form. The analysis targets on the majority rule model evolving on Regular Random (RRN), Erdős–Rényi, and Watts-Strogatz (small-world) networks. We construct the coarse-grained bifurcation diagrams with respect to the switching probability and we show how the connectivity distribution may result to symmetry breaking of the underlying macroscopic dynamics.

1. INTRODUCTION

Symmetry breaking of majority rule dynamics has been associated to phenomena such as herd behaviour under panic (Altshuler et al., 2005), the emergence of cooperation (Pacheco et al., 2009) dynamics and public opinion formation (Ianni & Corradi, 2002). For individualistic/ stochastic models whose dynamics evolve on complex networks, the extraction of closed coarse-grained models in the form of ordinary (ODEs) and/ or partial-integro-differential (PIDEs) equations is not an easy task. Due to the stochastic, nonlinear nature, multi-scale character and complexity of the network-deployed interactions, such equations are simply not available, or overwhelming difficult to derive. Without the existence of such models, what is usually done for analysis purposes is simple brute-force simulations: starting from different initial conditions run in time and average over many ensembles to get the required statistics. Even if we try to exploit the tools of Statistical Mechanics in order to derive some closures, these are just approximations that may introduce biases in the modelling and therefore in the analysis of the actual emergent dynamics.
This imposes a major impediment in our ability to analyse in a rigorous way the system's behaviour. In order to systematically analyse the way symmetry breaking influences the emergent dynamics of the majority rule model evolving on complex networks we exploit the Equation-Free framework (Kevrekidis et al., 2003) bypassing the construction of explicit coarse-grained models. In particular, we construct the coarse-grained bifurcation diagrams of the basic majority rule model, for Regular Random, Erdős–Rényi, and Watts-Strogatz (small-world) networks (Newman, 2003) with respect to the switching probability and analyze the stability of the computed stationary solutions.
The paper is organized as follows: in section 2, for the completeness of the presentation, we give a brief review of some basic complex networks. In section 3 we describe the majority rule model while in section 4 we prove analytically how the connectivity degree of regular random graphs governs the symmetry and the symmetry breaking of the solutions of the corresponding mean field models. In section 5, we show how the Equation-Free framework can be exploited to perform systems level tasks on heterogeneous networks. In section 6 we present

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the results of the numerical analysis, constructing the coarse-grained bifurcation diagrams of the majority rule dynamics as these obtained by exploiting the Equation-free approach. We conclude in section 7.

2. A BRIEF REVIEW OF SOME BASIC COMPLEX NETWORKS

A network is described as a graph, i.e. a pair of a set $G=(V,E)$, where V is the set of vertices (or, as otherwise called, nodes) and E is the set of the edges (or as otherwise called, links). A graph can be described by an adjacency matrix $A=[a_{ij}]$, ($i,j=1,2,...,N$) whose elements are defined as follows: if there is an link between nodes i and j then set $a_{ij}=1$; otherwise set $a_{ij}=0$. If the network is unidirectional (which means that node i is connected with node j and vice versa) then $a_{ij}=a_{ji}$.
Usually, the properties of the networks are studied in terms of the following three basic measurements:

(A) The characteristic path length L, defined as the mean value of all the shortest paths between any two nodes, i.e.

$$L=\frac{\sum d_{i\leftrightarrow j}}{N(N-1)} \tag{1}$$

where $d_{i\leftrightarrow j}$ is the shortest path between i and j nodes and N is the size of the network. The characteristic path length L is a global property of a network indicating the average number of steps needed to reach any two nodes.

(B) The density of cliques (Watts and Strogatz, 1998; Albert and Barabasi, 2002). Shortly speaking, a clique is a complete subgraph within a graph. An effective way to measure the "cliqueness" of a network is through the clustering coefficient c_i of the node i. This is defined as follows: let k_i be the degree of node i, i..e. the number of edges connected to node i. If the number of possible edges between the k_i neighbours (or the total number of possible triangles) of node i is $\frac{k_i(k_i-1)}{2}$ and the number of edges that really exist is E_i (i.e. the number of existing triangles), then the clustering coefficient c_i is defined as

$$c_i=\frac{2E_i}{k_i(k_i-1)} \tag{2}$$

The clustering coefficient C of the whole network is defined as the mean value of the clustering coefficients c_i of every node.

(C) The degree distribution $P(k)$ which gives the fraction of nodes with exactly k edges connected to it. Characteristic examples of almost symmetric-around the mean value of the degrees-distributions are the Erdős–Rényi and Watts and Strogatz type networks; scale-fee networks are characterized from power-law distributions (Albert & Barabasi, 2002; Newman, 2003).

Networks are usually separated into four major categories (Strogatz, 2001): (a) regular, (b) random, (c) small-world and (d) scale-free. Lattices, usually with four or eight neighbours for each node and rings, with the same connectivity degree for each node are characteristic representatives of regular networks. Regular networks are characterized by both big mean lengths and clustering coefficients contrary to the randomly connected graphs, such as RRN and the Erdős–Rényi, which are characterized by small values for both quantities (Newman, 2003). Over the last years it has been demonstrated that regularity or complete randomness are not characteristic of real-world networks. On the other hand, small-world and scale-free networks pertain to the structure of many problems ranging from neuroscience and epidemiology to the social sciences, internet and communication (Strogatz, 2001). Here, our analysis is focused on three celebrated types of networks, namely RRN, the Erdős–Rényi (ER) and Watts and Strogatz (WS) networks. The Erdős–Rényi networks (Albert and Barabasi, 2002) are constructed in the following simple way: in a population of N nodes it is assumed that each node can be connected with the other $N-1$ nodes with a probability p. This means that a node has an equal probability, say p, to be connected with every other node in the network. This type of network exhibits three distinct phases: (i) for relatively small values of the connection probability p ($p<\frac{1}{N}$) the network consists of many isolated subgraphs, (ii) for $p>\frac{1}{N}$ there is a giant cluster and (iii) for $p\geq\frac{\ln N}{N}$ almost the whole network is complete (Albert and Barabasi, 2002). The degree distribution follows the binomial law reading:

$$P(k)=\binom{N-1}{k}p^k(1-p)^{N-1-k} \tag{3}$$

This distribution is symmetric around the mean value $\bar{k}$ of the degrees which is given by $\bar{k}\cong pN$. As $N\to\infty$ the binomial distribution can be approximated by the Poisson distribution

$$P(k)=\frac{e^{-\bar{k}}\left(\bar{k}\right)^k}{k!} \tag{4}$$

The mean length follows the law $L\sim\frac{\ln N}{\ln\bar{k}}$ and the clustering coefficient is given by $C=p=\frac{\bar{k}}{N}$.

In their seminal paper, Watts and Strogatz, (1998), constructed a graph which can interpolate between a regular and a random one. It can be shown, that this type of network is characterized by the “small- world” property according to which the smallest path between two nodes is very short despite the large scale property of the underlying topology. This phenomenon is also known as the “six degrees of separation”. These networks combine the small world property (small characteristic path length) appearing in random graphs, with high clustering coefficient appearing in regular lattices.

In order to construct the network, Watts and Strogatz used a “rewiring” algorithm which can be described as follows: start with a ring network with K neighbours per node ($K/2$ left and right) ($N>>K>>\ln N$); with probability p cut the existent edge between a node and its first nearest neighbour (in a clock or counter clock wise sense) and rewire the edge with a random selected node. Self-connections or duplicate connections are not allowed. Repeat this process for every node regarding the first nearest neighbours. Repeat the same procedure for the second nearest neighbours etc. For $p=0$ the initial ring is invariant, while for $p=1$ the network is completely random. For the intermediate values $0<p<1$ the network interplays between a regular and a random network. When $p\to0$, then $L\approx\frac{N}{2K}$ and $C\approx\frac{3}{4}$ (which are typical values of a ring) and when $p\to1$, then $L\approx\frac{\ln N}{\ln K}$ and $C\approx\frac{K}{N}$ (which are typical values of a random network). Nevertheless, there is a broad band of values for p where $L \approx L_{Random}$ and $C > C_{Random}$, resulting to highly clustered networks but with a small characteristic path (Watts and Strogatz, 1998). The degree distribution of a WS network exhibits the following distribution

$$P(k) = \sum_{n=0}^{f(k,K)} \binom{K/2}{n} (1-p)^n p^{K/2-n} \frac{(pK/2)^{k-K/2-n}}{(k-K/2-n)!} e^{-pK/2} \tag{5}$$

where $k \geq K/2$, where $f(k,K) = \min\{k - K/2, K/2\}$ and p is the rewiring probability, K is the number of the initial neighbors for every node, corresponding to $p = 0$ ($K/2$ into left and right).

Scale-free property characterizes the structure of many real-world networks including biological systems and the world wide web (Newman, 2003; Strogatz, 2001). Such networks are usually characterized by power law degree distributions i.e. $P(k) = ck^{-\gamma}$, or on a logarithmic scale: $\ln P(k) = \ln c - \gamma \ln k$. Such networks are created through the "rich get richer" preferential connection algorithm (Albert and Barabasi, 2002).

3. THE MAJORITY RULE MODEL

Each neuron is labeled as i ($i = 1,2,...,N$), and its state gets two values: the value "1" if is it activated and the value "0" if it is not. We describe the state of the $i-th$ neuron in time t with the function $a_i(t) \in \{0,1\}$. Let $\Lambda(i)$ be the set of the neighbors (i.e. the neurons connected to $i-th$ neuron, with self loop included). Also consider the summation

$$\sigma_i(t) = \sum_{j \in \Lambda(i)} a_i(t)$$

which gives the number of activated neighbors of the i-*th* neuron. At each time step each neuron interacts with its neighboring neurons, and changes its state-value according to the following stochastic way (Kozma et al., 2005; Spiliotis and Siettos, 2011):

1. An inactive neuron becomes activated with probability ε, if $\sigma_i(t) \leq \frac{k_i+1}{2}$ (k_i is the degree of the $i-th$ neuron). If $\sigma_i(t) > \frac{k_i+1}{2}$ the neuron becomes activated with probability $1-\varepsilon$.
2. An activated neuron becomes inactive with probability ε, if the $\sigma_i(t) > \frac{k_i+1}{2}$. If $\sigma_i(t) \leq \frac{k_i+1}{2}$ the neuron becomes inactivate with probability $1-\varepsilon$.

ε takes values in the interval $(0, 0.5)$.

4. SYMMETRY AND SYMMETRY BREAKING OF THE MAJORITY RULE MODEL EVOLVING ON COMPLETE RANDOM NETWORKS WITH CONSTANT CONNECTIVITY

Let's start by assuming a complete network with constant, odd, connectivity degree $k = 5$ (self loop is included). Let d_t be the density of activate individuals at time t, defined as:

$$d_t = \frac{N_{act}}{N} \tag{6}$$

According to the mean field perspective, the time evolution of the density is given according to the following equation

$$d_{t+1} = f(d_t) \tag{7}$$

In the case of a constant degree distribution, say equal to five, the right-hand side of eq. (10) can be also written as

$$f(d,\varepsilon) = (1-\varepsilon) f_1(d) + \varepsilon f_2(d) \tag{8}$$

where

$$f_1(d) = d^5 + 5d^4(1-d) + 10d^3(1-d)^2,\ f_2(d) = 10d^2(1-d)^3 + 5d(1-d)^4 + (1-d)^5$$

For different values of the parameter ε the fixed points of equation (1) are given by solving the fixed point map

$$d = f(d,\varepsilon) \Leftrightarrow d - f(d,\varepsilon) = 0 \Leftrightarrow G(d,\varepsilon) = 0 \tag{9}$$

where $G(d,\varepsilon) = d - f(d,\varepsilon)$.

It can be shown that for a network with constant, odd degree distribution $k = 2l - 1$. Then the fixed point equation (9) for a constant ε has symmetric solutions with respect to the horizontal line $d = \frac{1}{2}$ (figure 1A).

For a complete network with arbitrary even $k = 2l$ constant connectivity degree, the time evolution of the density reads

$$d_{t+1} = f_{2l}(d_t, \varepsilon)$$

with

$$f_{2l}(d,\varepsilon) = (1-\varepsilon)\sum_{i=0}^{l-1}\binom{2l}{i} d^{2l-i}(1-d)^i + \varepsilon\sum_{i=l}^{2l-1}\binom{2l}{i} d^{2l-i}(1-d)^i$$

The above equation can be written as

$$f_{2l}(d,\varepsilon) = (1-\varepsilon) f_{1,2l}(d) + \varepsilon f_{2,2l}(d) + \varepsilon\binom{2l}{l} d^l (1-d)^l$$

where $f_{1,2l}(d) = \sum_{i=0}^{l-1}\binom{2l}{i} d^{2l-i}(1-d)^i$ and $f_{2,2l}(d) = \sum_{i=l}^{2l-1}\binom{2l}{i} d^{2l-1-i}(1-d)^i$

The two parts $f_{1,2l}(d), f_{2,2l}(d)$ are symmetric with respect to $d = \frac{1}{2}$. However due to the perturbation term $\varepsilon\binom{2l}{l} d^l (1-d)^l$ the function $f_{2l}(d,\varepsilon)$ loses its symmetry (figure 1B).

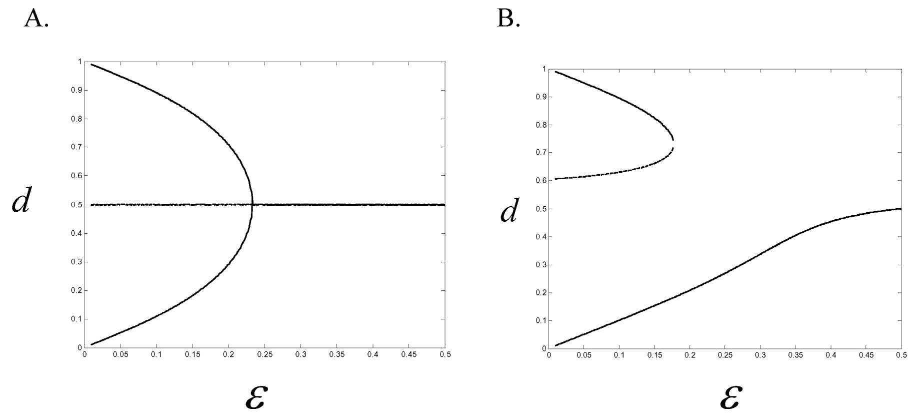

Fig. 1. Bifurcation diagram of the density of activated neurons with respect to the activation probability ε ***A****. constructed under the mean field approximation for a RRN with constant degree distribution equal to 5,* ***B****. constructed under the mean field approximation for a RRN with constant degree distribution equal to eight. There is a symmetry breaking of the Pitchfork Bifurcation. Dotted lines correspond to unstable solutions, while solid ones to stable stationary solutions.*

5. THE EQUATION-FREE APPROACH

The Equation-free approach can be used to bypass the need for extracting explicit continuum models in closed form (Makeev et al., 2002; Gear et al., 2002; Kevrekidis et al., 2003; Siettos et al., 2003). The main assumption of the framework is that macroscopic models in principle exist and close in terms of a few coarse-grained variables, which are usually the first moments of the underlying microscopic distributions; all the other higher-order moments become very-fast in the macroscopic time, functionals of the lower-order ones. What the methodology does, is to provide these closures "on demand" in a strict computational manner. A caricature of the method is described in the following steps:

(a) Choose the coarse-grained statistics, say $\boldsymbol{x}$, for describing the emergent behavior of the system and an appropriate representation for them (for example the mean value of the underlying evolving distribution).
(b) Choose an appropriate lifting operator μ that maps $\boldsymbol{x}$ to a detailed distribution $\boldsymbol{U}$ on the network. (For example, μ could make random state assignments over the networks which are consistent with the densities).
(c) Prescribe a continuum initial condition at a time t_k, say, $\boldsymbol{x}_{t_k}$.
(d) Transform this initial condition through lifting to N_r consistent individual-based realizations $\boldsymbol{U}_{t_k} = \mu \boldsymbol{x}_{t_k}$.
(e) Evolve these N_r realizations for a desired time *T*, generating the $\boldsymbol{U}_{t_{k+1}}$, where $t_k = kT$.
(f) Obtain the restrictions $\boldsymbol{x}_{t_{k+1}} = \aleph \boldsymbol{U}_{t_k}$.

The above steps, constitute the so called *coarse timestepper*, which, given an initial coarse-grained state of the system $\boldsymbol{x}_{t_k}$ at time t_k reports the result of the integration of the model over the network after a given time-horizon *T (at time* t_{k+1}), i.e.

$\boldsymbol{x}_{t_{k+1}} = \boldsymbol{\Phi}_T(\boldsymbol{x}_{t_k}, \boldsymbol{p})$, where $\boldsymbol{\Phi}_T : R^n x R^m \rightarrow R^n$ having $\boldsymbol{x}_{\boldsymbol{k}}$ as initial condition.

6. NUMERICAL ANALYSIS RESULTS

The numerical analysis was obtained using networks of $N = 10000$ neurons. We performed a coarse-grained analysis for RRN, Erdős–Rényi, Watts-Strogatz (small-world) networks. The bifurcation diagrams, with respect to the activation probability parameter ε , were constructed exploiting the Equation-free framework as described in the previous section. For our illustrations, our coarse-grained variable was the density d of the active individuals. At time t_0, we created N_r different distribution realizations consistent with the macroscopic variable d denoting the density of activated neurons. The coarse timestepper is constructed as the map:

$$d_{t+1} = \Phi_T(d, \varepsilon) \tag{10}$$

The derived coarse-grained bifurcation diagrams are depicted in figures 2 and 3. These are obtained using the detailed stochastic majority-rule simulator as a black-box timestepper and wrapping around it the Newton-Raphson iterative procedure in order to find the fixed points of the map (10). In the figures, dotted lines correspond to unstable solutions, while solid ones to stable stationary solutions. Figure 2 illustrates the coarse-grained bifurcation diagram when the underlying structure follows an Erdős–Rényi topology constructed using a with connectivity probability $p = 0.0008$. Figure 3 shows the derived coarse-grained bifurcation diagram in the case of a Watts-Strogatz network constructed with a rewiring probability $p = 0.2$ starting from a ring lattice with eight neighbours per node (four left and four right). As it is shown in figures 2, 3 the heterogeneity in the connectivity distribution results to a symmetry breaking of the stationary solutions.

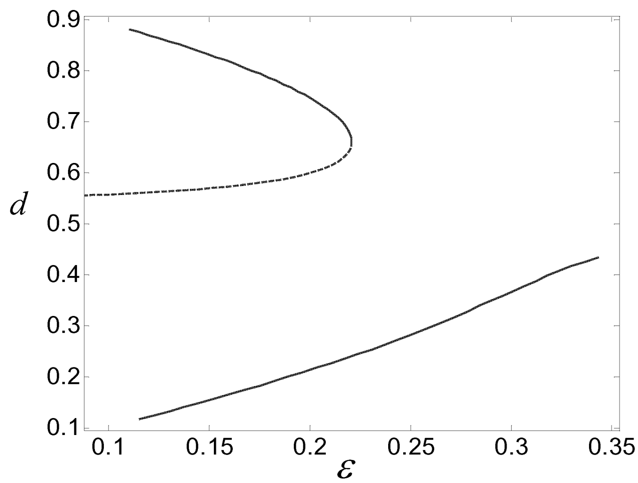

Figure 2: Coarse-grained bifurcation diagram of the density of activated neurons vs. ε for an Erdős–Rényi network constructed using $p = 0.0008$.

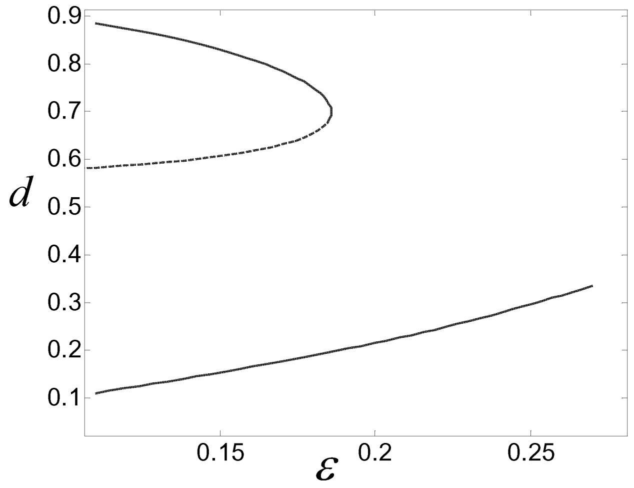

Fig. 3: Coarse-grained bifurcation diagram of the density of activated neurons vs. ε for a Watts-Strogatz network constructed with a rewiring probability $p = 0.2$ starting from a ring lattice with eight neighbours per node (four left and four right).

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