AIDIC *Conference Series*, Vol. 9, 2009
Editor Sauro Pierucci
DOI: 10.3303/ACOS0909022

ISBN 978-88-95608-10-5 ISSN 2036-5969

ON THE RELATIONSHIP BETWEEN SOME TRANSFORMATIONS FOR POSITIVE SIGNOMIAL TERMS

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Global optimization of mixed integer nonlinear programming (MINLP) problems containing signomial terms is in many cases a difficult task, and many different approaches to solve these problems have been devised. In Westerlund (2005) a method, where a convex relaxation of the original problem is obtained by approximating single-variable transformations with piecewise linear functions (PLFs), is described. What transformations are applicable depends on the sign of the signomial terms. For positive terms power transformations (PTs) (Lundell *et al.*, 2007) and the exponential transformation (ET) (Lundell and Westerlund, 2008) can be used. In Lundell and Westerlund (2009) it was shown that the ET always results in a tighter underestimator than the negative power transformation (NPT). These results are here extended to also include the positive power transformation (PPT); we will show that the PPT is better than the NPT under certain conditions, and also describe the relationship between the PPT and the ET.

1. INTRODUCTION

A special group of functions appearing in mathematical models of many processes is the signomial functions. A signomial function is defined as the sum of signomial terms, which in turn are products of power functions multiplied with a real constant, *i.e.*,

$$\sigma(\boldsymbol{x}) = \sum_j c_j \prod_i x_i^{p_{ji}}.$$

The definition of a signomial function provides a very general formulation incorporating many different types of functions; for example, bi- and trilinear terms, as well as all polynomials, can all be regarded as special cases of signomial functions.

Global optimization of problems containing signomials is often a difficult task and most methods available today uses convex underestimators in an iterative manner to find the global optimum. Since convex envelopes for general signomial terms are not known, other types of underestimators must be used. One global optimization method is the SGO algorithm described in Lundell (2009), which can be used to find the optimal solution to problems of the type

$$\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, x_2, \dots, x_I),\\
\text{subject to} \quad & \mathbf{A}\boldsymbol{x} = \boldsymbol{a},\\
& \mathbf{B}\boldsymbol{x} \le \boldsymbol{b},\\
& \boldsymbol{g}(\boldsymbol{x}) \le \mathbf{0},\\
& \boldsymbol{q}(\boldsymbol{x}) + \boldsymbol{\sigma}(\boldsymbol{x}) \le \mathbf{0},
\end{aligned}$$

where the functions f, g and q are convex, the functions σ are signomial and all variables occurring in the signomial terms are positive. In the SGO algorithm convex underestimators for the nonconvex signomial terms in σ

Please cite this article as: Lundell A. and Westerlund T., (2009), On the relationship between some transformations for positive signomial terms, AIDIC Conference Series, 09, 187-196 DOI: 10.3303/ACOS0909022

are obtained through single-variable power or exponential transformations. When approximating the relation between the original variables and the transformation variables with PLFs, the whole MINLP problem can be written in a convex relaxed form, which can be solved using any convex MINLP solver. By utilizing information obtained from the solution, the overestimation is made tighter by iteratively adding more breakpoints to the PLFs, *i.e.*, improving the approximations, until the global optimal solution is found.

The transformation process of the nonconvex constraints can be decomposed into the following steps:

$$q_m(\boldsymbol{x}) + \sigma_m(\boldsymbol{x}) \leq 0 \xrightarrow{(i)} q_m(\boldsymbol{x}) + \sigma_m^C(\boldsymbol{x}, \boldsymbol{X}) \leq 0 \xrightarrow{(ii)} q_m(\boldsymbol{x}) + \sigma_m^C(\boldsymbol{x}, \widehat{\boldsymbol{X}}) \leq 0.$$

In step *(i)* the nonconvex terms are transformed to a convex form using single-variable transformations of the form $x_i = T_{ji}(X_{ji})$, indicating a relation between the original variables x_i and the transformation variables X_{ji}. The problem is still nonconvex, since this relation must also be included in the transformed MINLP problem. However, in step *(ii)* the nonlinear relation is approximated with PLFs and the resulting overestimated problem will be convex in relaxed extended space containing the original variables and the new transformation variables.

The convergence properties of the function $q_m(\boldsymbol{x}) + \sigma_m^C(\boldsymbol{x}, \widehat{\boldsymbol{X}}_{\boldsymbol{k}})$ in the convexified signomial constraint, can be summarized as:

$$q_m(\boldsymbol{x}) + \sigma_m^C(\boldsymbol{x}, \widehat{\boldsymbol{X}}_{\boldsymbol{1}}) \leq \cdots \leq q_m(\boldsymbol{x}) + \sigma_m^C(\boldsymbol{x}, \widehat{\boldsymbol{X}}_{\boldsymbol{k}}) \leq q_m(\boldsymbol{x}) + \sigma_m^C(\boldsymbol{x}, \widehat{\boldsymbol{X}}_{\boldsymbol{k+1}}) \leq \cdots \leq q_m(\boldsymbol{x}) + \sigma_m(\boldsymbol{x}),$$

where $\widehat{X}_k$ corresponds to the values obtained from the PLFs when using the breakpoints in the k-th step of the algorithm. Furthermore, provided that the new breakpoints are chosen in a certain way, *e.g.*, by adding the mid-point of the largest interval between two existing breakpoints, the following is true

$$\lim_{k\to\infty} q_m(\boldsymbol{x}) + \sigma_m^C(\boldsymbol{x}, \widehat{\boldsymbol{X}}_{\boldsymbol{k}}) = q_m(\boldsymbol{x}) + \sigma_m(\boldsymbol{x}).$$

Depending on the sign of the signomial terms, different transformation schemes may be used. However, in this paper only positive terms are considered; for negative terms, see Lundell and Westerlund (2008) or Lundell *et al.* (2009). A positive nonconvex signomial term can be convexified by applying the single-variable exponential transformation (ET) $x_i = e^{X_i}$ or single-variable power transformation (PT) $x_i = {X_i}^{Q_i}$ to the variables having positive powers p_i.

Definition 1. The exponential transformation (ET). A convex underestimator for a positive signomial term is obtained when applying the transformations $x_i = e^{X_i}$ to all variables x_i with positive powers p_i, as long as the relation between the transformation variable X_i and the original variable x_i, *i.e.*, $X_i = \ln x_i$, is approximated with a PLF $\widehat{X}_i$.

The power transformations for positive signomial terms can be split into two different types depending on whether all transformation powers are kept negative or not. In the first type of transformation, all powers in the convexified signomial term will be negative.

Definition 2. The negative power transformation (NPT). A convex underestimator for a positive signomial term is obtained when applying the transformations $x_i = {X_i}^{Q_i}, Q_i < 0$, to all variables x_i with positive powers p_i, as long as the relation between the transformation variable X_i and the original variable x_i, *i.e.*, $X_i = x_i^{1/Q_i}$, is approximated with a PLF $\widehat{X}_i$.

The other alternative is to allow one variable to have a positive power in the convexified term. In this case, however, an additional condition on the transformation powers must be introduced.

Definition 3. The positive power transformation (PPT). A convex underestimator for a positive signomial term is obtained when applying the transformations $x_i = {X_i}^{Q_i}$ to all variables x_i with positive powers p_i, as long as the relation between the transformation variable X_i and the original variable x_i, *i.e.*, $X_i = x_i^{1/Q_i}$, is approximated with a PLF $\hat{X}_i$. All transformation powers Q_i are negative except for one (the k-th one), which is larger than or equal to one. Furthermore, the sum of the powers in the transformed term must be larger than or equal to one, *i.e.*,

$$\sum_{i:p_i>0} p_i Q_i + \sum_{i:p_i<0} p_i \geq 1.$$

As stated, the inverse transformations must be approximated using PLFs. These can be expressed in many different ways, *e.g.*, using binary variable or special ordered set formulations. See Westerlund (2005) or Lundell and Westerlund (2009) for details.

Now an example where a convex underestimator for a nonconvex signomial function of one variable is obtained using the different types of transformations.

Example 1. The nonconvex function $f(x) = -8x + 0.05x^3 + 25x^{0.5}$, $1 \leq x \leq 7$, consists of two convex terms as well as the nonconvex signomial term $25x^{0.5}$. Transforming the nonconvex term using the ET and the PTs and replacing the inverse transformation with PLFs, gives the convexified and underestimated functions

$$f(x, \hat{X}_E) = -8x + 0.05x^3 + 25e^{0.5\hat{X}_E} \quad \text{and} \quad f(x, \hat{X}_P) = -8x + 0.05x^3 + 25\hat{X}_P^{0.5Q}.$$

The PLFs $\hat{X}_E$ and $\hat{X}_P$ are given by the following expressions (where $\underline{x} = 1$ and $\overline{x} = 7$)

$$\hat{X}_E(x) = \ln \underline{x} + \frac{\ln \overline{x} - \ln \underline{x}}{\overline{x} - \underline{x}}(x - \underline{x}) \quad \text{and} \quad \hat{X}_P(x) = \underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x})$$

if only one interval is considered. If more breakpoints are used in the PLFs, then, *e.g.*, the following formulation, using so-called special ordered sets, can be used:

$$x = \sum_{k=1}^{K} x_k w_k, \qquad \hat{X} = \sum_{k=1}^{K} X_k w_k, \qquad \sum_{k=1}^{K} w_k = 1, \qquad 0 \leq w_k \leq 1,$$

where $\{w_k\}_{k=1}^{K}$ is a special ordered set of type 2, *i.e.* at most two elements in the set may be nonzero and all nonzero elements must be consecutive. The parameters X_k are the values the inverse transformation assumes at the K consecutive points $x_1 < x_2 < \cdots < x_K$, *i.e.*, $X_k = \ln x_k$ or $X_k = x_k{}^{1/Q}$. Illustrations of the convex underes-

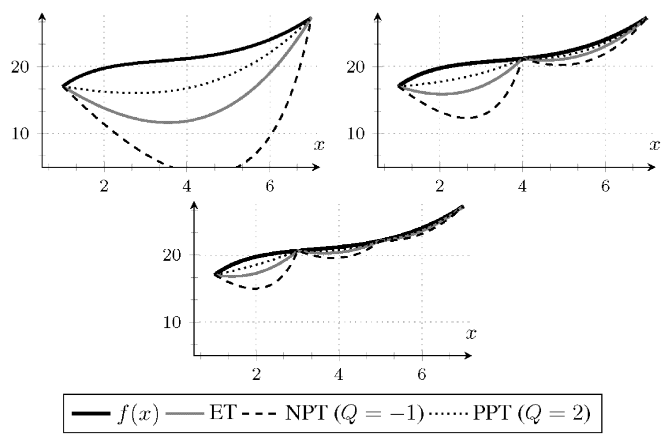

Figure 1. The piecewise convex underestimators in Example 1, when using one, two and three intervals

timators are provided in Figure 1 for different number of breakpoints in the PLFs. The underestimator is actually piecewise convex in the illustration, however, since we have increased the variable space with the variables used in the PLFs, the underestimator is actually convex in the extended space.

2. THEORETICAL RESULTS

The convex underestimators obtained from the different types of transformations, and also for different values of the parameter Q in the PTs, give rise to different underestimation errors when they are used to transform the nonconvex signomial terms. To be able to determine which transformations perform the best in certain cases, some theoretical results regarding the transformations are given in this section.

2.1. Underestimation relationships for the single-variable transformations

In this section we will show some new results regarding the relation between the PTs and the ET when applied to an individual power function x^p. This is a continuation of the work in Lundell and Westerlund (2009) where the following theorem was shown.

Theorem 1. The ET always gives a tighter convex underestimator when applied to a signomial term of one variable, *i.e.*, a the power function x^p, than the NPT regardless of the power Q used in the NPT.

The results in Theorem 1 can be extended to also include the PPT:

Theorem 2. A single-variable PPT always gives a tighter convex underestimator than both the ET and NPT.

Proof. This proof is similar to the proof for Theorem 1, which can be found in Lundell and Westerlund (2009). When the power function x^p is transformed by the transformations $x = X_P^Q, Q \geq 1$ (PPT) and $x = e^{X_E}$ (ET) and approximated with the PLFs $\hat{X}_P$ and $\hat{X}_E$ respectively, the claim that the PPT is a tighter underestimator than the ET is equivalent to

$$(\hat{X}_P^Q)^p \geq \left(e^{\hat{X}_E}\right)^p \overset{p>0}{\Longrightarrow} \hat{X}_P^Q \geq e^{\hat{X}_E}.$$

We now consider a one-step PLF only, so the previous inequality is equivalent to

$$\left(\underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}\left(x - \underline{x}\right)\right)^Q \geq \exp\left(\ln \underline{x} + \frac{\ln \overline{x} - \ln \underline{x}}{\overline{x} - \underline{x}}\left(x - \underline{x}\right)\right).$$

By modifying this expression and introducing $z = \left(\frac{\overline{x}}{\underline{x}}\right)^{1/Q}$ and $\lambda = \frac{x - \underline{x}}{\overline{x} - \underline{x}}$, we receive the equivalent form $\lambda z \geq z^\lambda + \lambda - 1$, which can easily be proven to be true whenever $Q \geq 1$ and $\underline{x} \leq x \leq \overline{x}$ by searching for the minimum value of the function $f(z) = \lambda z - z^\lambda - \lambda + 1$. Thus, the PPT gives a tighter convex underestimator than the ET when applied to individual variables. Furthermore, since we know from Theorem 1 that the ET gives better approximations than the NPT, it is obvious that the same is true for the PPT. It is also clear that the approximations are only equal in the endpoints $\underline{x}$ and $\overline{x}$. ■

Theorem 3. For the piecewise linear approximations $\hat{X}_P$ (PPT and NPT) and $\hat{X}_E$ (ET), the resulting convex underestimations of the PTs when the power Q goes to plus or minus infinity is equal to that of the ET, *i.e.*,

$$\lim_{Q \to \pm\infty} \hat{X}_P^Q = e^{\hat{X}_E}.$$

Proof. The claim was proved for negative values of Q in Lundell and Westerlund (2009). The proof is, however, valid for positive values of Q as well. ■

2.2. Underestimation relationships for general positive signomial terms

Using the previous theorems for the transformations of the individual power functions, some results for when underestimating entire positive signomial terms can be obtained.

Theorem 4. The ET always gives a tighter underestimator than the NPT when applied to a general nonconvex positive signomial term.

Proof. Since it is true that the ET gives a tighter underestimator of each individual power function, it is also true that the ET gives a tighter underestimator of the whole signomial term. ■

A corresponding result for the PPT and the NPT can also be found, however, now additional conditions on the transformation powers Q must be included.

Theorem 5. The PPT gives a tighter underestimator than the NPT when applied to a general nonconvex positive signomial term as long as the powers used in the single-variable PTs with negative powers $Q_{i,P}$ and $Q_{i,N}$ in the PPT and NPT respectively fulfill $Q_{i,P} \leq Q_{i,N}$.

Proof. The claim is equivalent to

$$\hat{s}_P(\boldsymbol{x}) \geq \hat{s}_N(\boldsymbol{x})$$

where $\hat{s}_P(\boldsymbol{x})$ and $\hat{s}_N(\boldsymbol{x})$ are the convex underestimators of the general nonconvex signomial term

$$s(\boldsymbol{x}) = \prod_{i \in I_+} x_i^{p_i} \cdot \prod_{i \in I_-} x_i^{p_i}, \qquad I_+ = \{i: p_i > 0\}, \qquad I_- = \{i: p_i < 0\},$$

obtained when transforming the term using PPT and NPT respectively. When applying the PTs, only the variables x_i: $i \in I_+$ need to be transformed. When transforming this term using the PPT, all PTs use negative transformation powers, *i.e.*, $Q_{i,P} < 0,\ i \in I_+$, except for the k-th one, for which $Q_{k,P} \geq 1, k \in I_+$. Under these conditions, the following holds

$$\begin{aligned} \hat{s}_P(\boldsymbol{x}) &= \prod_{i \in I_+} \hat{X}_{i,P}^{p_i Q_{i,P}} \cdot \prod_{i \in I_-} (x_i)^{p_i} = \hat{X}_{k,P}^{p_k Q_{k,P}} \cdot \prod_{i \in I_+ \setminus \{k\}} \hat{X}_{i,P}^{p_i Q_{i,P}} \cdot \prod_{i \in I_-} x_i{}^{p_i} \\ &\geq \hat{X}_{k,N}^{p_k Q_{k,N}} \cdot \prod_{i \in I_+ \setminus \{k\}} \hat{X}_{i,N}^{p_i Q_{i,N}} \cdot \prod_{i \in I_-} x_i{}^{p_i} = \prod_{i \in I_+} \hat{X}_{i,N}^{p_i Q_{i,N}} \cdot \prod_{i \in I_-} (x_i)^{p_i} = \hat{s}_N(\boldsymbol{x}), \end{aligned}$$

where the facts that the single-variable PT with positive power is always tighter than the single-variable PT with negative power, as well as that $Q_{i,\mathrm{P}} \leq Q_{i,\mathrm{N}}$ holds for all $i \neq k$ has been used. ■

Although the PPT always results in a tighter convex underestimator than the ET when applied to an individual power function, as the following theorem shows, it is not generally true for the case when a PPT is applied to a signomial term of more than one variable.

Theorem 6. Neither the PPT nor the ET gives a tighter underestimator in the whole domain when applied to a nonconvex signomial term of more than one variable.

Proof. For a general nonconvex signomial term

$$s(\boldsymbol{x}) = \prod_{i \in I_+} x_i^{p_i} \cdot \prod_{i \in I_-} x_i^{p_i}, \qquad I_+ = \{i: p_i > 0\}, \qquad I_- = \{i: p_i < 0\},$$

only the variables x_i: $i \in I_+$ must be transformed. When using the ET all of the variables with positive powers are transformed using single-variable ETs and when using the PPT one variable (with arbitrary index $k \in I_+$) is transformed using a single-variable positive PT and the rest (with indices $i \in I_+ \setminus \{k\}$) using single-variable negative PT. We know from Theorems 1 and 2 that

$$\hat{X}_{k,PT}^{Q_k p_k} \geq \left(\exp \hat{X}_{k,ET}\right)^{p_k} \qquad \text{and} \qquad \forall i \in I_+ \setminus \{k\}\colon\ \hat{X}_{i,PT}^{Q_i p_i} \leq \left(\exp \hat{X}_{i,ET}\right)^{p_i}.$$

when the transformation variables $X_{i,PT}$ and $X_{i,ET}$ have been replaced with the PLFs $\hat{X}_{i,PT}$ and $\hat{X}_{i,ET}$. Now, take two points $\boldsymbol{x}^* = (x_1^*, \dots, x_N^*)$ and $\boldsymbol{x}^\# = (x_1^\#, \dots, x_N^\#)$ such that:

$$x_i^*\colon \begin{cases} i = k\colon & \underline{x}_i < x_i^* < \overline{x}_i, \\ i \in I_+ \setminus \{k\}\colon & x_i^* = \underline{x}_i \vee x_i^* = \overline{x}_i\,, \\ i \in I_-\colon & \underline{x}_i \leq x_i^* \leq \overline{x}_i, \end{cases} \qquad x_i^\#\colon \begin{cases} i = k\colon & x_i^\# = \underline{x}_i \vee x_i^\# = \overline{x}_i, \\ i \in I_+ \setminus \{k\}\colon & \underline{x}_i < x_i^\# < \overline{x}_i, \\ i \in I_-\colon & \underline{x}_i \leq x_i^\# \leq \overline{x}_i. \end{cases}$$

Then, for the point $\boldsymbol{x}^*$ the following is true

$$\begin{aligned} \hat{s}_P(\boldsymbol{x}^*) &= \prod_{i\in I_+} \hat{X}_{i,PT}^{p_i Q_i} \cdot \prod_{i\in I_-} (x_i^*)^{p_i} = \hat{X}_{k,PT}^{p_k Q_k} \cdot \prod_{i\in I_+\setminus\{k\}} \left(\exp \hat{X}_{i,ET}\right)^{p_i} \cdot \prod_{i\in I_-} (x_i^*)^{p_i} \\ &> \left(\exp \hat{X}_{k,ET}\right)^{p_k} \prod_{i:i\in I_+\setminus\{k\}} \left(\exp \hat{X}_{i,ET}\right)^{p_i} \cdot \prod_{i\in I_-} (x_i^*)^{p_i} = \hat{s}_E(\boldsymbol{x}^*), \end{aligned}$$

since the underestimations resulting from the PPT always lie above those of the ET when $\underline{x}_i < x_i^* < \overline{x}_i$ and the NPT and the ET are equal at the endpoints, *i.e.*, when $x_i^* = \underline{x}_i \vee x_i^* = \overline{x}_i$. Therefore, the PPT gives a tighter underestimation than the ET in the chosen point . Correspondingly, for the point $\boldsymbol{x}^\#$ the following is true

$$\begin{aligned} \hat{s}_P(\boldsymbol{x}^\#) &= \prod_{i\in I_+} \hat{X}_{i,PT}^{p_i Q_i} \cdot \prod_{i\in I_-} \left(x_i^\#\right)^{p_i} = \left(\exp \hat{X}_{k,ET}\right)^{p_k} \cdot \prod_{i\in I_+\setminus\{k\}} \hat{X}_{i,PT}^{p_i Q_i} \cdot \prod_{i\in I_-} \left(x_i^\#\right)^{p_i} \\ &< \left(\exp \hat{X}_{k,ET}\right)^{p_k} \cdot \prod_{i:i\in I_+\setminus\{k\}} \left(\exp \hat{X}_{i,ET}\right)^{p_i} \cdot \prod_{i\in I_-} \left(x_i^\#\right)^{p_i} = \hat{s}_E(\boldsymbol{x}^\#). \end{aligned}$$

Thus, since both $\hat{s}_P(\boldsymbol{x}^*) > \hat{s}_E(\boldsymbol{x}^*)$ and $\hat{s}_P(\boldsymbol{x}^\#) < \hat{s}_E(\boldsymbol{x}^\#)$ are true, neither the PPT nor the ET gives tighter underestimations in the whole domain. ■

Since the underestimators are continuous and neither of the underestimators is tighter in the whole domain, there must be parts of the domain where they are equal. This region is given in the following corollary:

Corollary 1. The regions where the PPT and the ET are equal, the PPT is tighter than the ET, and the ET is tighter than the PPT consists of the parts of the domain where the following expressions are respectively true

$$\prod_{i\in I_+} \hat{X}_{i,PT}^{p_i Q_i} = \prod_{i\in I_+} e^{p_i \hat{X}_{i,ET}}, \qquad \prod_{i\in I_+} \hat{X}_{i,PT}^{p_i Q_i} > \prod_{i\in I_+} e^{p_i \hat{X}_{i,ET}}, \qquad \prod_{i\in I_+} \hat{X}_{i,PT}^{p_i Q_i} < \prod_{i\in I_+} e^{p_i \hat{X}_{i,ET}}, \qquad I_+ = \{i\colon p_i > 0\}.$$

3. ILLUSTRATIONS AND EXAMPLES

Some illustrations of the convex underestimators arising from different transformations are now provided.

Example 2. The nonconvex function $f(x) = -8x + 0.05x^3 + 25x^{0.5}$, $1 \leq x \leq 7$, from Example 1, is now used to illustrate how the values of the power Q in the PTs impact the resulting underestimator. Transforming the nonconvex term using the ET and the PTs and replacing the inverse transformation with PLFs, gives the convexified and underestimated functions

$$f\left(x, \hat{X}_E\right) = -8x + 0.05x^3 + 25e^{0.5\hat{X}_E} \quad \text{and} \quad f\left(x, \hat{X}_P\right) = -8x + 0.05x^3 + 25\hat{X}_P^{0.5Q}.$$

The underestimators resulting from the ET and the PTs (with different values on the transformation power Q) are plotted in Figure 2. From the figure, it can be seen that the PPT and NPT gets closer to the ET when the power Q gets larger or smaller respectively, as stated in Theorem 3.

Example 3. The ET, PPT and NPT are now applied to the function $f(x,y) = xy$. Using the transformation $x = e^{X_E}$ in the ET, and the transformation $x = X_P^Q$ for the PTs, the transformed terms will then take the forms:

ET: $xy \rightarrow e^{X_E}e^{Y_E}$, PPT: $xy \rightarrow X_P^{Q_{x,1}}Y_P^{Q_{y,1}}$, NPT: $xy \rightarrow X_P^{Q_{x,2}}Y_P^{Q_{y,2}}$.

In the PPT, the transformation powers $Q_{x,1}$ and $Q_{y,1}$ must be chosen such that one is positive, the other negative, and their sum is greater than or equal to one, *e.g.*, $Q_{x,1} = 2$ and $Q_{y,1} = -1$. The conditions on the transformation powers in the NPT is that they are both negative, *e.g.*, $Q_{x,2} = Q_{y,2} = -1$. Furthermore, the inverse transformations $X_E = \ln x$, $Y_E = \ln y$ and $X_P = x^{1/Q_x}, Y_P = y^{1/Q_y}$ must be replaced by the PLFs $\hat{X}_E, \hat{Y}_E$ and $\hat{X}_P$, $\hat{Y}_P$. For the variable x in one step on the interval $[\underline{x}, \overline{x}]$ these PLFs are given as

$$\hat{X}_E(x) = \ln \underline{x} + \frac{\ln \overline{x} - \ln \underline{x}}{\overline{x} - \underline{x}}(x - \underline{x}) \quad \text{and} \quad \hat{X}_P(x) = \underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x}).$$

Illustrations of the underestimators are provided in Figure 3 when $x \in [1,7]$. In Figure 4, the region where the PPT is a tighter underestimator than the ET is shown. This region has been obtained using the expression in Corollary 1.

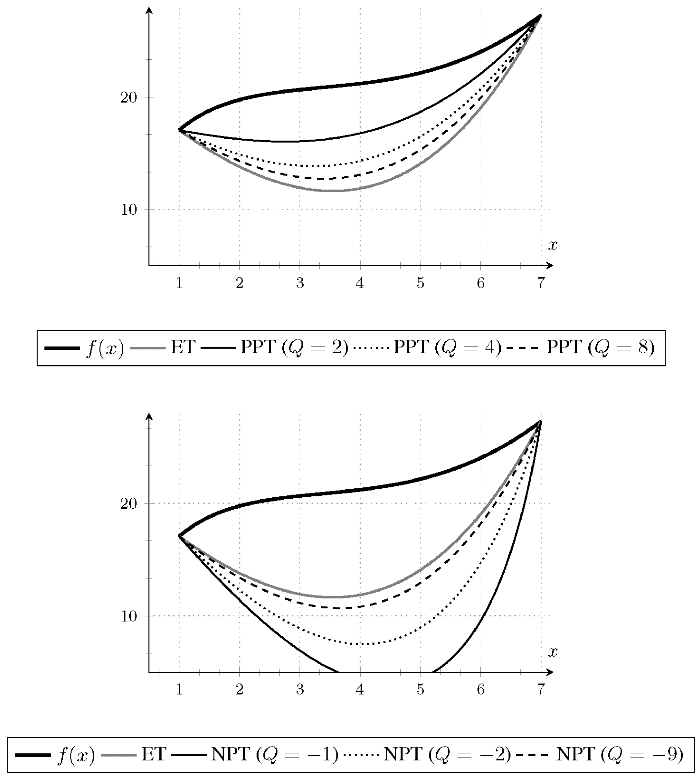

Figure 2. An illustration of how the convex underestimators for the function f(x) in Example 2 change as the transformation power Q is altered.

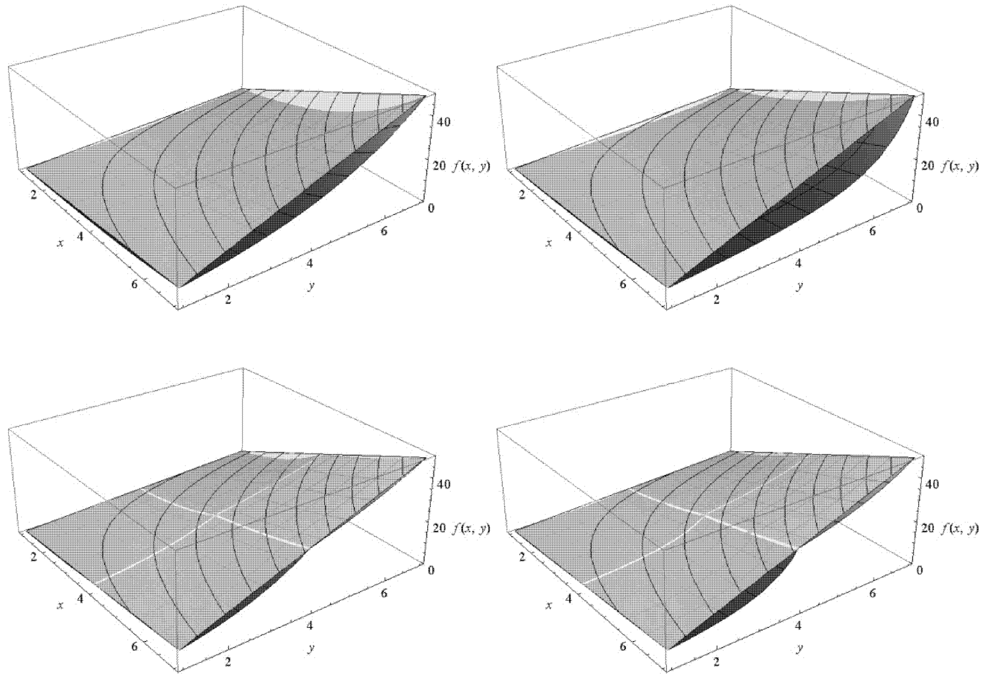

Figure 3. The convex underestimators resulting from the ET (left) and the PPT (right) for the function f(x,y) in Example 3. The pictures below are with the extra breakpoints x=y=4 added to the PLFs.

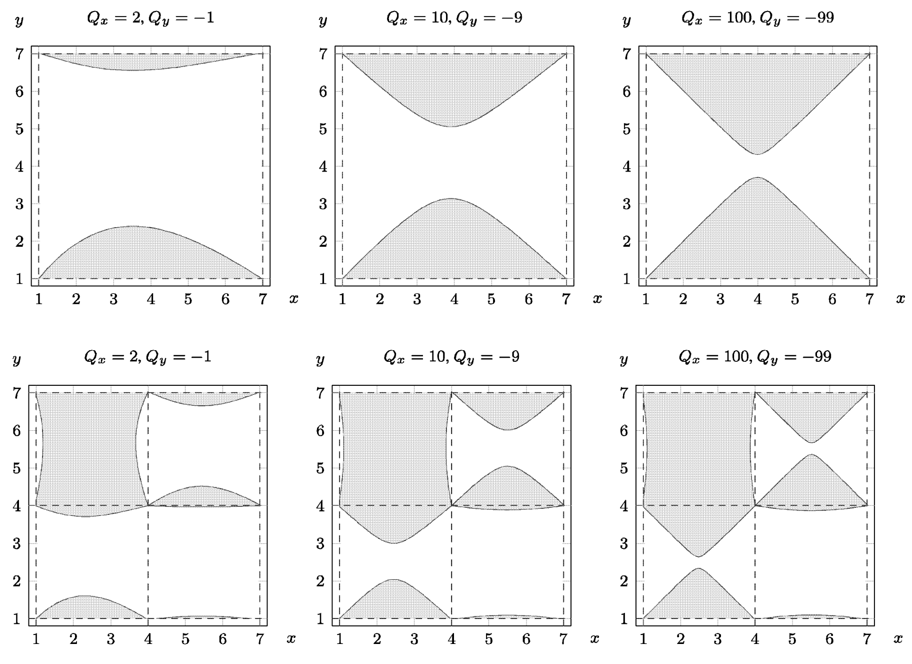

Figure 4. The shaded area indicates where the PPT gives a tighter underestimator than the ET for the function f(x,y) in Example 3. The values of the transformation powers are indicated in the figure. The pictures above is without extra breakpoints in the PLFs, and the pictures below are with extra breakpoints at x=y=4. The area has been obtained using the expression in Corollary 1.

4. CONCLUSIONS

In this paper, it was shown that the PPT gives a tighter convex underestimator than both the NPT and the ET when applied to an individual variable. Generally, however, the PPT applied to signomial terms of more than one variable only gives a tighter underestimator than the ET in parts of the domain. An implicit expression showing in which parts of the domain the ET is superior is easy to deduce (Corollary 1), however, at this stage it is still not known whether explicit conditions exists.

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