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# Control of Energy Dissipation in a Strong Nonlinear, Damped and Periodically Excited System

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In the biomass-fired boilers intensive deposit layer on the heating surface is formed. For deposit elimination various cleaning procedures are applied. The mechanical effect of cleaning is based on vibration of the system. The efficiency of the procedure depends on the parameters of the tube-deposit vibrating system. In the paper the system is assumed to be a truly nonlinear oscillator with linear damping which is excited with a periodical cleaning force. The exact nonlinear resonance for the system is determined and exact steady states of the oscillator are computed. Special attention is given to the influence of damping properties of the deposit on the system vibration and on the energy dissipation due to mismatch between the excitation and damping. For determination of the energy change an approximate analytic solving procedure is developed. The method represents the adopted version of the time variable amplitude and phase procedure. As an example, the tube-deposit system described with damped Duffing oscillator excited with Jacobi elliptic function is considered. Analytically obtained solution is compared with numerical one. The results are in good agreement and prove the correctness of the suggested model. Finally, the generalized mathematical model of the tube-slag oscillatory system gives prediction of nonlinear resonant vibration caused by periodical cleaning force.

## 1. Introduction

Nowadays, significant attention is given to effective energy production by firing of biomass. Usually, agricultural residual and herbaceous material, wood or bark, human and animal waste, contaminated or industrial biomass (Masia et al., 2007) are fired. During combustion of the solid biomass flues it is found that ash is built up and the un-burnt materials form a thick deposit layer on heating surfaces (Kleinhans et al., 2017) that decreases or even stops the heat transfer in the boiler. Various methods for deposit shedding are developed Zbogar et al. (2009) and more recently Laxminarayan et al. (2017). Most often cleaning procedures are applied. For all of these methods it is common that they cause vibration in the system. The vibration of tubes gives a contribution to tube cleaning due to shaking of the build-up material. If the excitation force is higher than the adhesion force between the deposit and tube the deposit removes from the heating surface. However, one may wonder which intensity of the cleaning force to utilize for cleaning of the surface without damaging it. In praxis, this force is determined by the operator from case to case. To overcome this lack and to generalize the vibration problem, we introduce the mathematical model for vibration of the deposit-tube system. The model is a non-homogenous second order truly nonlinear differential equation (Cveticanin, 2018). In the paper the exact steady states of the oscillator are computed and analyzed. Vakakis and Blanchard (2018), developed a method for computing exact steady states for the damped Duffing oscillator. In this paper we extended the method for solving equations with strong nonlinearity of any order (integer or non-integer). For the condition when the restitution force in oscillator corresponds to the adhesion force between deposit and tube the nonlinear vibration response is determined. Amplitude and frequency of vibration depend on the excitation parameters but also on the damping property of the deposit. If the damping property of the deposit layer varies it causes changes in vibration. In the paper the perturbed model of the tube-deposit oscillating system is also considered. Approximate solution of the problem is obtained. It is concluded that the damping parameter seems to be the appropriate control parameter for optimization of the cleaning process.

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## 2. Periodically excited strong nonlinear oscillator

The tube with deposit excited with the external period force is modeled as one degree of freedom oscillatory system. Physical model is a mass – spring system. Mass is the sum of tube and deposit masses. The elastic force of the tube - deposit system is assumed to be a strong nonlinear deflection function. The damping force of the deposit is supposed to be the linear velocity function. Vibration is caused with the time periodical cleaning excitation force. Then, the non-dimensional equation of vibration is

$$\ddot{x} + c_\alpha x|x|^{\alpha-1} + b\dot{x} = F(t) \tag{1}$$

where $c_\alpha$ is the coefficient of rigidity, *b* is the damping coefficient, $\alpha > 1$ is the order of nonlinearity (any integer or non-integer) and *F*(*t*) is the excitation force. If in the system damping and excitation are omitted the equation (1) simplifies into

$$\ddot{x} + c_\alpha x|x|^{\alpha-1} = 0. \tag{2}$$

For initial conditions $x(0) = A$ and $\dot{x}(0) = 0$ the solution of Eq(2) is

$$x = Aca(\alpha, 1, \Omega t), \tag{3}$$

where *A* is the amplitude of vibration, *ca* is the *cosine* Ateb function (Cveticanin, 2018) and Ω is the frequency of the Ateb function which satisfies the relation

$$-\frac{2}{\alpha+1}\Omega^2 + c_\alpha A^{\alpha-1} = 0. \tag{4}$$

Adopting the procedure for steady state solution of strong nonlinear differential equations (Vakakis and Blanchard, 2018) it is obtained that for

$$F(t) = -F_0 sa(\alpha, 1, \Omega t), \tag{5}$$

where *sa* is the *sine* Ateb function and

$$F_0 = bA\Omega\frac{2}{\alpha+1}. \tag{6}$$

the relation (3) with (4) is the exact solution of (1).

*Remark*: Ateb function is the inverse Beta function. For $\alpha = 3$ it is transformed into Jacobi elliptic function and for $\alpha = 1$ in trigonometric harmonic function.

For the steady state motion (3) the excitation is with a periodical *sa* function with period (Cveticanin, 2018)

$$T = \frac{2\Pi}{\Omega}, \tag{7}$$

where Π = B(1/(α+1), ½) and B is the beta function. The *sa* function is the simplified version of a multi-harmonic force which is the sum of harmonic functions with various frequencies. In Figure 1a) the *sa* excitation function for Ω = 1 and various values of *α* is plotted. Analyzing Fig.1a) it is seen that the higher is the value of *α*, the longer is the period of the function *sa*. Using Eq(5) the considered model of the periodically excited and damped truly nonlinear oscillator is

$$\ddot{x} + c_\alpha x|x|^{\alpha-1} + b\dot{x} = -F_0 sa(\alpha, 1, \Omega t). \tag{8}$$

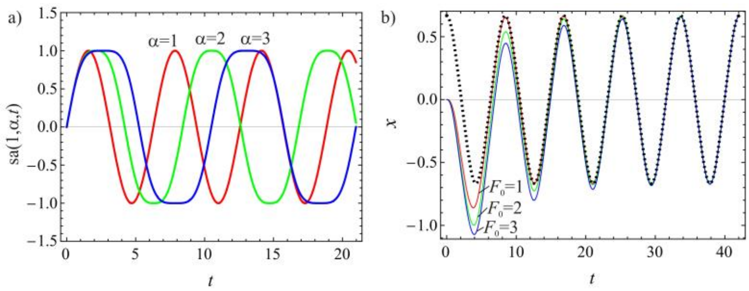


*Figure 1: a) The sa function for Ω=1 and α=1 (red line), α=2 (green line) and α=3 (blue line); b) x-t diagrams for $F_0$=1 (red line), $F_0$=2 (green line) and $F_0$=3 (blue line) and analytically obtained function (dotted line).*

For initial conditions

$$x(0) = 0, \qquad \dot{x}(0) = 0, \tag{9}$$

the steady state solution Eq(3) exists if conditions Eq(4) and Eq(6) are satisfied.

## 3. Steady state motion

According to the previous consideration it is obvious that for the excitation parameters $F_0$ and $\Omega$, and system parameters $b$ and $c_\alpha$, the Eq(1) has the steady state solution Eq(3) for

$$A = \left(\frac{2}{\alpha+1}\frac{\Omega^2}{c_\alpha}\right)^{\frac{1}{\alpha-1}}, \qquad b = F_0\left(\frac{c_\alpha}{\Omega^{\alpha+1}}\left(\frac{\alpha+1}{2}\right)^{\alpha}\right)^{\frac{1}{\alpha-1}}. \tag{10}$$

Namely, if the damping coefficient $b$ satisfies the relation $(10)_2$ the steady state vibration Eq(3) of the system is periodical and has the amplitude $(10)_1$. The steady state motion remains unperturbed if the damping coefficient is controlled according to $(10)_2$. Thus, if the frequency of excitation is constant and the intensity of excitation is varied, the damping coefficient has to be treated according to $(10)_2$ and the amplitude of vibration $(10)_1$ is unperturbed.

In Figure 1b) the numerical solution of

$$\ddot{x} + x|x| + \frac{9}{4}F_0\dot{x} = -F_0 sa(1,2,t), \tag{11}$$

is compared with analytical solution (3)

$$x = \frac{2}{3}ca(1,2,t) \tag{12}$$

It can be seen that there is no difference between the analytical and numerical solution. However, analysing relations (10) it is obvious that the steady state motion exists for constant intensity and variable frequency of excitation, if the damping coefficient is controlled due to $(10)_2$. Then, the amplitude of the steady state motion is also varied, according to $(10)_1$. In Figure 2 the amplitude of vibration and damping coefficient versus excitation frequency curves are plotted. The dimensionless amplitude $A_2/A_1$, frequency $\Omega_2/\Omega_1$ and damping coefficient $b_2/b_1$, are introduced. It is concluded that the amplitude of vibration depends on the excitation frequency: the higher is the value of the frequency, the higher is the amplitude of vibration. For the Duffing oscillator with cubic nonlinearity the amplitude – frequency relation is linear. For $1 < \alpha < 3$ the amplitude increase with frequency is slower, while for $\alpha > 3$ it is faster than for the boundary value $\alpha = 3$. The damping coefficient decreases by increasing the frequency of excitation. The decrease is faster for higher values of nonlinearity order $\alpha$.

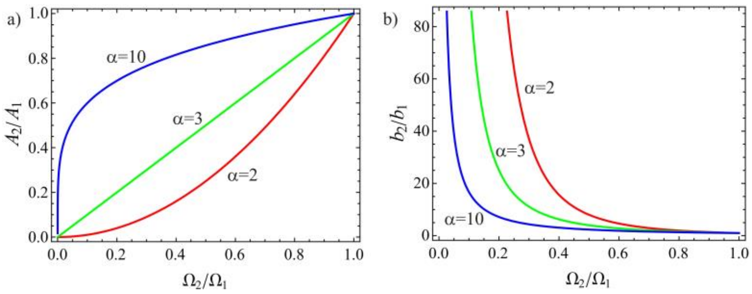


*Figure 2: a) amplitude-frequency and b) damping-frequency diagrams for: α =2 (red line), α =3 (green line) and α =10 (blue line).*

In Figure 3 the amplitude – excitation intensity and frequency – excitation intensity functions are plotted. $F_2/F_1$ is the dimensionless excitation intensity.

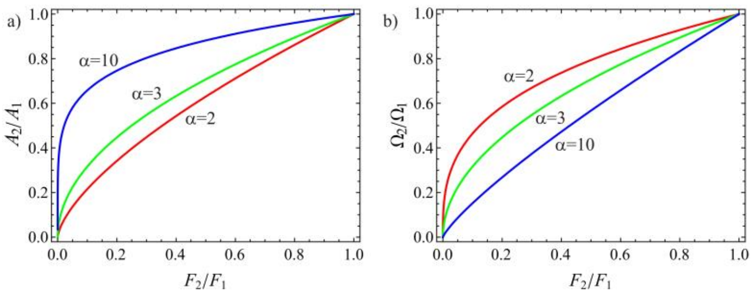


*Figure 3: a) amplitude-excitation and b) frequency-excitation diagrams for: α =2 (red line), α =3 (green line) and α =10 (blue line).*

From Fig.3 it is seen that both vibration parameters, the amplitude and the frequency, increase by increasing of the excitation intensity. The amplitude of vibration increases faster and the frequency of vibration slower if the order of nonlinearity $\alpha$ is higher. Thus, for $\alpha$ tends to infinity, the frequency – excitation intensity relation tends to be linear.

## 4. Energy dissipation rate

As it is mentioned in Section 2, the relation EQ(3) is the solution of the free undamped oscillator Eq(2) for initial conditions $x(0) = A$ and $\dot{x}(0) = 0$. The oscillator is a conservative system and the total energy of the system is constant

$$E = \frac{c_\alpha}{\alpha+1} A^{\alpha+1} = const. \tag{13}$$

The energy constant depends on the amplitude of vibration $A$. If for certain value of $A$ the excitation frequency $\Omega$ satisfies the relation $(10)_1$ and the excitation intensity or damping coefficient $b$ correspond to the relation $(10)_2$ the forced and damped nonlinear oscillator is dynamically equivalent to the free nonlinear undamped oscillator Eq(2) and oscillators have the same energy level

$$E = \frac{c_\alpha}{\alpha+1} A^{\alpha+1} = \frac{c_\alpha}{\alpha+1}\left(\frac{2}{\alpha+1}\frac{\Omega^2}{c_\alpha}\right)^{\frac{\alpha+1}{\alpha-1}} = \frac{c_\alpha}{\alpha+1}\left(\frac{\alpha+1}{2}\frac{F_0}{b\Omega}\right)^{\alpha+1}. \tag{14}$$

Expression Eq(14) describes the influence of the excitation force on the energy level of the oscillator if conditions Eq(10) are satisfied. By varying the excitation parameters $F_0$ and $\Omega$ the vibration energy is changing. If the excitation frequency is higher, the amplitude of vibration and the vibration energy are higher. The same conclusion is evident for higher value of the excitation force. By decreasing of the excitation parameters the vibration energy decreases, too. During decrease of the excitation parameters the energy dissipation occurs. If the initial energy is $E_1$ and the final energy for the decreased excitation $E_2$ the energy difference is $E_1 - E_2 > 0$. Dividing the energy difference with the initial energy $E_1$ a dimensionless coefficient $\varepsilon$, the so called 'dissipation rate' is defined as

$$\varepsilon = 1 - \frac{E_2}{E_1}. \tag{15}$$

The dissipation coefficient varies in the interval $\varepsilon = [0,1]$. For $\varepsilon = 0$ there is no dissipated energy, while for $\varepsilon = 1$ the whole energy is dissipated. Substituting (14) into (15) we have

$$\varepsilon = 1 - \left(\frac{A_2}{A_1}\right)^{\alpha+1} = 1 - \left(\frac{F_2}{F_1}\right)^{\alpha+1} = 1 - \left(\frac{\Omega_2}{\Omega_1}\right)^{2\frac{\alpha+1}{\alpha-1}} = 1 - \left(\frac{b_1}{b_2}\right)^{\alpha+1}. \tag{16}$$

The energy dissipation occurs if $A_1 > A_2$, $F_1 > F_2$, $\Omega_1 > \Omega_2$ and $b_1 < b_2$. The value of the dissipation rate depends on the amplitude, frequency, excitation or damping ratio. The higher is the ratio $A_1/A_2$, $F_1/F_2$ and $\Omega_1/\Omega_2$ the higher is the energy dissipation $\varepsilon$. By increasing the amplitude and frequency of vibration or increasing the frequency and intensity of the excitation force in comparison to the previous value, the energy dissipation rate will increase, too. In contrast, if the damping coefficient is decreased the damping rate is increased.

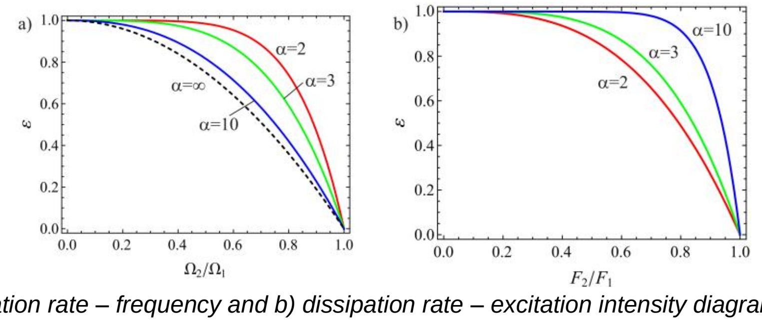


*Figure 4: a) dissipation rate – frequency and b) dissipation rate – excitation intensity diagrams for $\alpha$ =2 (red line), $\alpha$ =3 (green line), $\alpha$ =10 (blue line) and $\alpha = \infty$ (black dashed line)*

## 5. Vibration of the perturbed tube-deposit oscillatory system

In Figure 4 the energy dissipation rate versus excitation frequency and excitation intensity for various order of nonlinearity is plotted. For $\Omega_2/\Omega_1 < 1$ the higher the order of nonlinearity the smaller is the value of the energy dissipation rate. Besides, for higher order of nonlinearity the dissipation rate decrease is faster with excitation frequency than for smaller order of nonlinearity. If the of nonlinearity is extremely high the limit dispersion rate is

$$\varepsilon = 1 - \left(\frac{\Omega_2}{\Omega_1}\right)^2. \tag{17}$$

Analysing the relation Eq(16) and Fig.5b it is obvious that the energy dissipation – excitation intensity expression depends on the order of nonlinearity $\alpha$, too. The value of the dissipation rate, for the same rate of the excitation intensity ($F_2/F_1 < 1$), is smaller for smaller order of nonlinearity $\alpha$. By increasing the nonlinear property of the oscillator, whose excitation intensity is constant, the energy dissipation is increasing. The sensitivity of the system on the variation of the intensity of excitation is smaller if the nonlinearity is higher.

## 6. Vibration of the perturbed tube-deposit oscillatory system

To obtain the steady state motion Eq(3) of the oscillator Eq(1) the coefficient of damping $b$ has to satisfy the relation $(10)_2$. However, if there is a mismatch between the excitation and damping force due to variation of the damping coefficient $b_1$ in comparison to $b$, some perturbation of the steady state vibration would occur. Mathematical model of the perturbed system is

$$\ddot{x} + c_\alpha x|x|^{\alpha-1} + b\dot{x} = -F_0 sa(1, \alpha.\Omega t) - \varepsilon b_1 \dot{x}, \tag{18}$$

where $\varepsilon b_1 << 1$ is a small perturbation parameter. To solve (18) the approximate method based on time variable amplitude and phase is introduced. The solution of the system is assumed in the form

$$x = A(t) ca\big(\alpha, 1, \psi(t)\big), \quad \dot{x} = -\frac{2}{\alpha+1} A(t)\Omega(t) sa\big(1, \alpha, \psi(t)\big), \tag{19}$$

with

$$\dot{\psi}(t) = \Omega + \theta(t). \tag{20}$$

where $A(t)$, $\psi(t)$ and $\theta(t)$ are time variable functions. The first time derivative of $(19)_1$ is

$$\dot{x} = \dot{A}(t) ca\big(\alpha, 1, \psi(t)\big) - \frac{2}{\alpha+1} A(t)\Omega(t) sa\big(1, \alpha, \psi(t)\big) - \frac{2}{\alpha+1} A(t)\dot{\theta}(t) sa\big(1, \alpha, \psi(t)\big) \tag{21}$$

Comparing $\text{Eq}(19)_2$ and Eq(21) the constraint follows

$$\dot{A} ca - \frac{2}{\alpha+1} A\dot{\theta} sa = 0 \tag{22}$$

where for simplification $ca = ca(\alpha,1,\Omega t)$, $sa = sa(1,\alpha,\Omega t)$, $A = A(t)$, $\Omega = \Omega(t)$, $\theta = \theta(t)$. The time derivative of $\text{Eq}(19)_2$ is

$$\ddot{x} = -\frac{2}{\alpha+1}\dot{A}\Omega sa - \frac{2}{\alpha+1} A\dot{\Omega} sa - \frac{2}{\alpha+1} A\Omega\dot{\theta} ca^\alpha - \frac{2}{\alpha+1} A\Omega^2 ca^\alpha \tag{23}$$

Substituting (19) and (23) into (18) we obtain

$$-\frac{2}{\alpha+1}(\dot{A}\Omega + A\dot{\Omega}) sa - \frac{2}{\alpha+1} A\Omega\dot{\theta} ca^\alpha = \varepsilon b_1 \frac{2}{\alpha+1} A\Omega sa \tag{24}$$

Using the derivative of Eq(4)

$$-2\dot{\Omega} A + (\alpha-1)\Omega\dot{A} = 0 \tag{25}$$

the Eq(24) transforms into

$$\dot{A}\Omega sa - \frac{2}{\alpha+1} A\Omega\dot{\theta} ca^\alpha = \varepsilon b_1 \frac{2}{\alpha+1} A\Omega sa \tag{26}$$

Relations Eq(22) and Eq(26) correspond to Eq(18) and are the two first order differential equations of motion in new variables $A$ and $\theta$. After some modification we have

$$\dot{A}\Omega = -\varepsilon b_1 \frac{2}{\alpha+1} A\Omega sa^2, \quad \frac{2}{\alpha+1} A\Omega\dot{\theta} = -\varepsilon b_1 \frac{2}{\alpha+1} A\Omega saca \tag{27}$$

To find the exact solution of Eq(27) is impossible. The averaging of Eq(27) over the period of vibration

$$T = \frac{2\Pi}{\Omega} = \frac{2B(\frac{1}{\alpha+1}, \frac{1}{2})}{\Omega} \tag{28}$$

Is suggested. Averaged functions are

$$\langle sa^2 \rangle = \frac{1}{T}\int_0^T sa^2\, dt = \frac{\alpha+1}{\alpha+3}, \quad \langle sa\, ca \rangle = \frac{1}{T}\int_0^T sa\, ca\, dt = 0 \tag{29}$$

and averaged differential equations are

$$\dot{A} = -\varepsilon b_1 \frac{2}{\alpha+3} A, \quad \dot{\theta} = 0 \tag{30}$$

Integrating Eq(30) and using the steady state amplitude the motion of the oscillator is

$$x = A exp\left(-\varepsilon b_1 \frac{2t}{\alpha+3}\right) ca(\alpha, 1, \Omega t), \tag{31}$$

i.e. after substituting $(10)_1$

$$x = \left(\frac{2}{\alpha+1}\frac{\Omega^2}{c_\alpha}\right)^{\frac{1}{\alpha-1}} exp\left(-\frac{2\varepsilon b_1 t}{\alpha+3}\right) ca(\alpha, 1, \Omega t) \tag{32}$$

where $b$ satisfies the relation Eq(10)$_2$. Analysing the relation Eq(32) it is concluded that the additional damping term $\varepsilon b_1$ causes the vibration to decrease. The velocity of the amplitude variation depends on the order of nonlinearity: the higher is the order of nonlinearity the amplitude decrease is slower than for the linear case. If the nonlinearity is sufficiently large, the amplitude decrease is negligible: the small damping variation has no influence on the steady state motion of the oscillator. In Figure 5 the numerically obtained solution of Eq(18) is compared with analytically obtained one Eq(32).

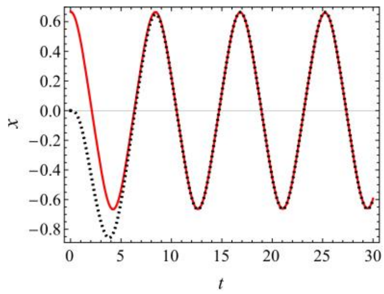


*Figure 5: x-t diagrams obtained: analytically (red line) and numerically (dotted black line)*

It can be concluded that the analytically obtained solution is on the top of the numerical one. The difference between solutions is negligible for small perturbation parameter $\varepsilon b_1$ and short time interval.

## 7. Conclusions

The paper is the pioneering investigation in mathematical formulation of the cleaning of tube-deposit system of a boiler caused by vibration. The vibrating tube-slag system is assumed as a one degree of freedom oscillator where the external periodical excitation force acts. Due to nonlinear properties of the system the mathematical model is a second order differential equation with the nonlinearity is of any order (integer or non-integer). The excitation is described with a periodical Ateb function. For the system the nonlinear resonant vibration is determined. An analytical procedure is developed for computing of the exact steady states of the oscillator. For the certain relation between the adhesion force in the tube-deposit system and the restitution force the amplitude of vibration is obtained. Varying the parameters of the excitation function the oscillator changes its steady states. During this process the energy dissipation occurs. In the paper the energy dissipation rate is introduced. Based on the values of this rate the sensitivity of the system on the variation of the excitation parameters is analysed. It is concluded that the oscillators with small order of nonlinearity are less sensitive on frequency of excitation change and the steady state motion remains almost with the same amplitude.
The perturbation of the vibration of the tube-deposit system occurs if the damping property of the deposit layer varies. In the paper approximate solution of the perturbed equation is obtained. Analytically obtained results are compared with numerical ones and are in good agreement. It is concluded that due to damping variation a transient motion to a new steady state motion occurs. The higher the order of nonlinearity of the system, the elimination of the perturbation due to variation of the damping coefficient is faster.

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