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Constrained Robust Model Predictive Control Based on Polyhedral Invariant Sets by Off-line Optimization

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This paper proposes a fast robust model predictive control using polyhedral invariant sets for uncertain polytopic discrete-time systems. A sequence of nested polyhedral invariant sets corresponding to a sequence of state feedback gains is constructed off-line. Thus, most of the computational burdens are moved off-line. At each sampling time, when the measured state lies between two adjacent polyhedral invariant sets, a state feedback gain is calculated by solving a linear programming based on linear interpolation between two pre-computed state feedback gains. The controller design is illustrated with an example. The simulation results showed that the proposed algorithm provides a better control performance while on-line computation is still tractable as compared to previously reported algorithms.

1. Introduction

Model predictive control (MPC) is a control technique that optimizes future behaviour of a process by using a process model. Robust model predictive control (RMPC) is a specific type of MPC which explicitly includes model uncertainty in the problem formulation. RMPC has been applied in a wide variety of application areas such as control of a tabular heat exchanger (Bakošová and Oravec, 2012) and distillation column (Martin et al., 2013). In RMPC, all possible state trajectories are restricted to lie in the invariant set constructed, so robust stability of the system can be guaranteed. Although the polyhedral invariant set is well-known to have some advantages over the ellipsoidal invariant sets such as better handling of asymmetric constraints and enlargement of stabilizable region (Pluymers et al., 2005), the ellipsoidal invariant set is usually used in RMPC formulation due to its relatively low on-line computational complexity. Recently, an off-line RMPC algorithm using polyhedral invariant sets has been developed by Bumroongsri and Kheawhom (2012b). The on-line computational complexity is reduced by constructing off-line a sequence of polyhedral invariant sets corresponding to a sequence of pre-computed state feedback gains. At each sampling instant, the smallest polyhedral invariant set containing the measured state is determined and the corresponding state feedback gain is implemented to the process. Thus, all of the computational burdens are moved off-line. However, the conservativeness is obtained because the control law implemented at each time step is only an approximation of the true optimal control law. Moreover, the input discontinuities caused by a switching between state feedback control laws are occurred. Therefore, the algorithm requires constructing a large number of polyhedral invariant sets, hence large data storage, in order to improve the control performance and reduce the input discontinuities.

In this paper, we present a fast RMPC using polyhedral invariant sets that requires very small computation complexity and data storage. A sequence of nested polyhedral invariant sets corresponding to a sequence of state feedback gains is constructed off-line. At each sampling instant, when the measured state lies between two adjacent polyhedral invariant sets, the real-time control law is calculated by solving a computationally low-demanding linear programming that is based on linear interpolation between two pre-computed state feedback gains.

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Notation: For a matrix A, A^T denotes its transpose, A^{-1} denotes its inverse. I denotes the identity matrix. For a vector x, $x(k/k)$ denotes the state measured at real time k, $x(k+i/k)$ denotes the state at prediction time $k+i$ predicted at real time k. The symbol $*$ denotes the corresponding transpose of the lower block part of symmetric matrices.

2. Problem formulation

The model considered here is the following linear time varying (LTV) system with polytopic uncertainty

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $x(k)$ is the state of the plant, $u(k)$ is the control input and $y(k)$ is the plant output. We assume that

$$[A(k), B(k)] \in \Omega, \Omega = Co\{[A_1, B_1], [A_2, B_2], ..., [A_L, B_L]\} \tag{2}$$

where Ω is the polytope, Co denotes convex hull, $[A_j, B_j]$ are vertices of the convex hull. Any $[A(k), B(k)]$ within the polytope is a linear combination of the vertices such that

$$[A(k), B(k)] = \sum_{j=1}^{L} \lambda_j(k)[A_j, B_j],\ \sum_{j=1}^{L} \lambda_j(k) = 1, 0 \le \lambda_j(k) \le 1 \tag{3}$$

where $\lambda(k) = [\lambda_1(k), \lambda_2(k), ..., \lambda_L(k)]$ is the uncertain parameter vector. The aim of this research is to find the state feedback control law

$$u(k+i/k) = Kx(k+i/k) \tag{4}$$

that stabilizes (1) and achieves the following performance cost under the nominal model assumption

$$\min_{u(k+i/k),\ i\ge 0} J_\infty(k)\ , J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix} \tag{5}$$

where $\Theta > 0$ and $R > 0$ are symmetric weighting matrices, subject to input and output constraints

$$|u_h(k+i/k)| \le u_{h,\max}, h = 1, 2, 3, ..., n_u \tag{6}$$

$$|y_r(k+i/k)| \le y_{r,\max}, r = 1, 2, 3, ..., n_y \tag{7}$$

3. The proposed algorithm

In this section, a fast RMPC using polyhedral invariant sets is presented. A sequence of polyhedral invariant sets corresponding to a sequence of state feedback gains is constructed off-line. At each sampling instant, when the measured state lies between two adjacent polyhedral invariant sets, the real-time state feedback gain is calculated by linear interpolation between two pre-computed state feedback gains. The idea of the proposed algorithm is based on linear interpolation between two pre-computed state feedback gains to get the real-time state feedback gain that is as large as possible.

Algorithm 3.1
Off-line step 1: Choose a sequence of states $x_i, i \in \{1, 2, ..., N\}$ and solve the following problem to obtain the corresponding state feedback gains $K_i = Y_i G_i^{-1}$

$$\min_{Y_i, G_i, Q_{j,i}} \gamma_i \tag{8}$$

$$\text{s.t.} \begin{bmatrix} 1 & * \\ x_i & Q_{j,i} \end{bmatrix} \geq 0 \,, \forall j = 1,2,...,L \tag{9}$$

$$\begin{bmatrix} G_i + G_i^T - Q_{j,i} & * & * & * \\ \hat{A} G_i + \hat{B} Y_i & Q_{l,i} & * & * \\ \Theta^{\frac{1}{2}} G_i & 0 & \gamma_i I & * \\ R^{\frac{1}{2}} Y_i & 0 & 0 & \gamma_i I \end{bmatrix} \geq 0, \forall j = 1,2,...,L, \forall l = 1,2,...,L \tag{10}$$

$$\begin{bmatrix} G_i + G_i^T - Q_{j,i} & * \\ A_j G_i + B_j Y_i & Q_l \end{bmatrix} \geq 0 \,, \forall j = 1,2,...,L, \forall l = 1,2,...,L \tag{11}$$

$$\begin{bmatrix} X & * \\ Y_i^T & G_i + G_i^T - Q_{j,i} \end{bmatrix} \geq 0, \forall j = 1,2,...,L,\ X_{hh} \leq u_{h,\max}^2 ,\ h = 1,2,...,n_u \tag{12}$$

Note that the following condition must be satisfied

$$Q_{j,i}^{-1} - (A_k + B_k K_{i+1})^T Q_{j,i}^{-1} (A_k + B_k K_{i+1}) > 0,\ \exists j = 1,2,...,L, \forall i = 1,2,..,N, \forall k = 1,2,...,L \tag{13}$$

The optimization problem used to derive the state feedback gains in this step is based on the online RMPC controller proposed by Bumroongsri and Kheawhom (2012a). It is the minimization of upper bound of infinite horizon nominal cost performance. However, the output constraints are relaxed in this step in order to enlarge the stabilizable region and to reduce conservativeness. The output constraints and also input constraints are then properly taken into account during polyhedral invariant set construction in off-line-step 2. The condition (14) is used to assure robust stability satisfaction of a convex combination of K_i and K_{i+1}.

Off-line Step 2: Construct a sequence of polyhedral invariant sets $S_i = \{x / M_i x \leq d_i\}, i = 1,2,...,N$ corresponding to a sequence of pre-computed state feedback gains $K_i, i = 1,2,...,N$ by following the procedures of Bumroongsri and Kheawhom (2012b).

On-line Step 1: At each sampling instant, if the measured state lies between S_i and S_{i+1} , $i = 1,2,..., N-1$, implement $K = \lambda K_i + (1-\lambda) K_{i+1}$ to the process where λ is calculated by solving the following optimization problem

$$\min_{\lambda} \lambda \tag{14}$$

s.t.

$$u_{\min} \leq (\lambda K_i + (1-\lambda) K_{i+1}) x_k \leq u_{\max} \tag{15}$$

$$M_i (A_j + B_j (\lambda K_i + (1-\lambda) K_{i+1})) x_k \leq d_i, \forall j = 1,2,...,L \tag{16}$$

$$0 \leq \lambda \leq 1 \tag{17}$$

If the state lies in S_N , implement K_N to the process.

Since input and output constraints impose lesser and lesser limits on state feedback gains as the state converges to the origin, the norm of pre-computed state feedback gain increase from outer to inner polyhedral invariant sets $(K_i < K_{i+1}, i = 1,2,..., N-1)$. By minimizing λ at each control iteration, the real-time state feedback gain that is as large as possible is implemented to the process.

4. An example

Consider the application of our approach to the nonlinear two-tank system which is described by the following equation

$$\rho S_1 \dot{h}_1 = -\rho A_1 \sqrt{2gh_1} + u \tag{18}$$

$$\rho S_2 \dot{h}_2 = \rho A_1 \sqrt{2gh_1} - \rho A_2 \sqrt{2gh_2} \tag{19}$$

where h_1 is the water level in tank 1, h_2 is the water level in tank 2 and u is the water flow. The operating parameters are shown in table 1.

Table 1: The operating parameters of nonlinear two-tank system.

Operating parameters	Values
S_1	$2{,}500\ \text{cm}^2$
S_2	$1{,}600\ \text{cm}^2$
A_1	$9\ \text{cm}^2$
A_2	$4\ \text{cm}^2$
g	$980\ \text{cm/s}^2$
ρ	$0.001\ \text{kg/cm}^3$

Let $\overline{h_1} = h_1 - h_{1,eq}$, $\overline{h_2} = h_2 - h_{2,eq}$ and $\overline{u} = u - u_{eq}$ where subscript eq is used to denote the corresponding variable at equilibrium condition, the objective is to regulate $\overline{h_2}$ to the origin by manipulating $\overline{u}$. The input and output constraints are $|\overline{u}| \le 1.5\ \text{kg/s}$, $|\overline{h_1}| \le 13\ \text{cm}$, $|\overline{h_2}| \le 50\ \text{cm}$.

By evaluating the Jacobian matrix of (17) and (18) along the vertices of the constraints set, we have that all the solutions are also the solution of the following differential inclusion

$$\begin{bmatrix} \rho S_1 \dot{\overline{h}}_1 \\ \rho S_2 \dot{\overline{h}}_2 \end{bmatrix} \in \left(\sum_{j=1}^{4} p_j A_j \right) \begin{bmatrix} \overline{h}_1 \\ \overline{h}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \overline{u} \tag{20}$$

where $A_j, j = 1,\ldots,4$ are given by

$$A_1 = \begin{bmatrix} -\rho A_1 \sqrt{\dfrac{2g}{h_{1,\min}}} & 0 \\ \rho A_1 \sqrt{\dfrac{2g}{h_{1,\min}}} & -\rho A_2 \sqrt{\dfrac{2g}{h_{2,\min}}} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -\rho A_1 \sqrt{\dfrac{2g}{h_{1,\max}}} & 0 \\ \rho A_1 \sqrt{\dfrac{2g}{h_{1,\max}}} & -\rho A_2 \sqrt{\dfrac{2g}{h_{2,\min}}} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -\rho A_1\sqrt{\dfrac{2g}{h_{1,\min}}} & 0 \\ \rho A_1\sqrt{\dfrac{2g}{h_{1,\min}}} & -\rho A_2\sqrt{\dfrac{2g}{h_{2,\max}}} \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -\rho A_1\sqrt{\dfrac{2g}{h_{1,\max}}} & 0 \\ \rho A_1\sqrt{\dfrac{2g}{h_{1,\max}}} & -\rho A_2\sqrt{\dfrac{2g}{h_{2,\max}}} \end{bmatrix} \tag{21}$$

The discrete-time model is obtained by discretization of (18) and (19) using Euler first-order approximation with a sampling period of 0.1 s and it is omitted here for brevity. The tuning parameters are $\Theta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 0.01$. The proposed algorithm is compared with its online counterpart RMPC algorithm proposed by Bumroongsri and Kheawhom (2012a) and the off-line RMPC algorithm proposed by Bumroongsri and Kheawhom (2012b).

Figure 1 shows a sequence of four polyhedral invariant sets $S_i = \{x / M_i x \le d_i\}, i = 1,2,...,4$ constructed off-line. The sizes of polyhedral invariant sets decrease from S_1 to S_4 while the norm of state feedback gains increase from S_1 to S_4.

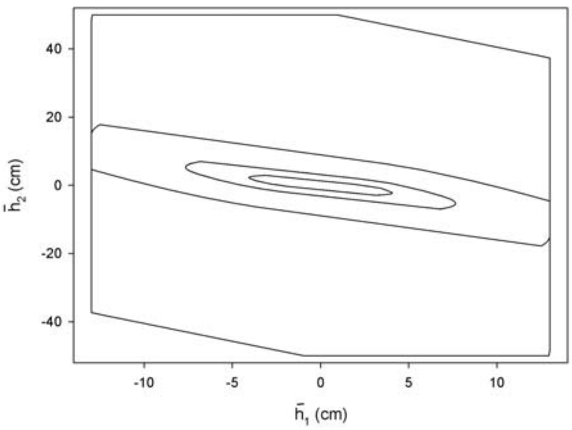

Figure 1: A sequence of four polyhedral invariant sets constructed off-line.

Figure 2 shows profiles of the water level in tank 2 (regulated output) and the water flow (control input) obtained by each algorithm. Algorithm 3.1 can steer the state to the origin faster that other algorithms. This is due to the fact that in the algorithm of Bumroongsri and Kheawhom (2012b), there is no interpolation between pre-computed state feedback gains. Consequently, the control law implemented at each time step is only an approximation of the true optimal control law.

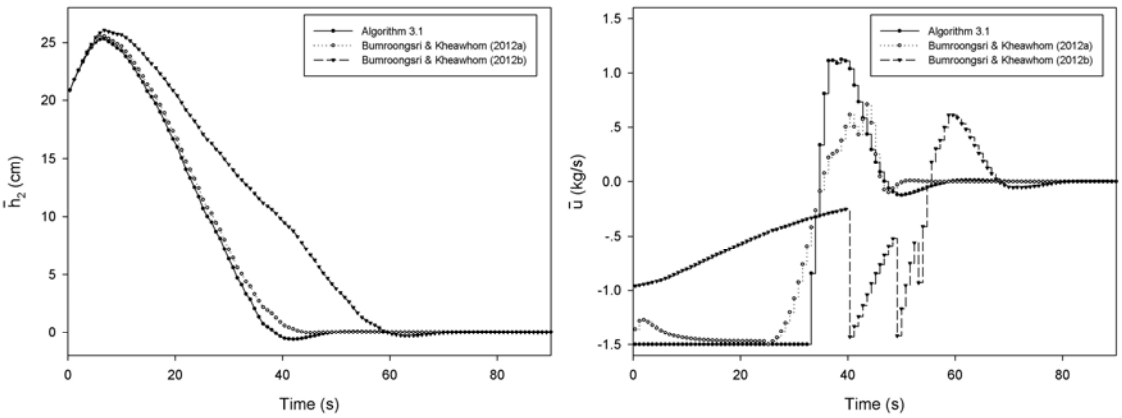

Figure 2: a) The water level in tank 2 (regulated output) and b) the water flow (control input).

The algorithm of Bumroongsri and Kheawhom (2012a) uses online optimization to compute a state feedback gain. However, the algorithm utilizes an ellipsoidal invariant set that is more conservative than a polyhedral invariant set. It is seen that by interpolation between pre-computed state feedback gains as proposed in algorithm 3.1, the control input variable becomes continuous, and the smooth response is obtained.

For each control iteration, the average on-line computational time required for algorithm 3.1 is as low as 0.001s, because all of the on-line optimization problems are formulated in the form of linear programming. The algorithm of Bumroongsri and Kheawhom (2012a) uses online optimization to compute a state feedback gain, and computational time of 0.3s is required for each sampling time.

5. Conclusions

In this paper, we have presented a fast RMPC using polyhedral invariant sets. A sequence of polyhedral invariant sets corresponding to a sequence of pre-computed state feedback gains is constructed off-line. At each sampling instant, the smallest invariant set containing the currently state measured is determined. A state feedback gain is calculated by solving a linear programming based on linear interpolation between two pre-computed state feedback gains associated with current invariant set determined and the adjacent smaller invariant set. The simulation results show that the proposed algorithms can achieve better control performance than existing algorithms including the off-line RMPC algorithm based on polyhedral invariant sets without interpolation, and its online counterpart RMPC algorithm based on ellipsoidal invariant sets.

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