AIDIC *Conference Series*, Vol. 10, 2011
Editor Sauro Pierucci
DOI: 10.3303/ACOS1110042

ISBN 978-88-95608-58-7 ISSN 2036-5969

SOME NOTES ON CONVEX RELAXATIONS

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Convex relaxations play an important role in many areas, especially in optimization and particularly in global optimization. In this paper we will consider some special, but fundamental, issues related to convex relaxation techniques in constrained nonconvex optimization. We will especially consider optimization problems including nonconvex inequality constraints and their relaxations. Finally, we will illustrate the results by some examples, including a problem connected to N-dimensional allocation.

1. INTRODUCTION AND MOTIVATION

In the area of optimization, different types of relaxation techniques are used. In this paper, we will focus on convex relaxations and especially on some properties related to these in connection to global optimization. Several global optimization methods are based on the principle of relaxing a nonconvex problem into convex subproblems and solving these iteratively. By using a branch and bound framework, a subdivision of the initial domain can be automated and the global optimal solution finally obtained (Floudas, 2000). Alternatively a reformulation framework, where the problem is convexified in an extended variable space, can be applied (Lundell et al., 2009). However, independently of the type of procedure used, it is important that the relaxations used when solving the subproblems are made as tight as possible.

In this paper, we will consider convex relaxations for optimization problems containing nonconvex inequality constraints. In such cases the nonconvex functions in the inequality constraints may be replaced by their convex envelopes. In several sources it is mentioned that by doing so, *e.g.* Tuy (1998), one will obtain the tightest convex relaxation for the nonconvex optimization problem. However, as we will show by focusing more carefully on the relaxations, the tightest convex relaxation for the functions defining the inequality constraints does not automatically result in the tightest convex relaxation for the problem at hand. Thus, even tighter convex relaxations for such optimization problems can be obtained.

2. PROBLEM FORMULATION

Let us consider the following constrained nonconvex optimization problem

$$\begin{aligned} &\text{minimize} && f(\boldsymbol{x}) && \boldsymbol{x} = (x_1, x_2, \ldots, x_N) \\ &\text{subject to} && g_m(\boldsymbol{x}) \le 0, && m = 1,2,\ldots,M, \end{aligned} \tag{1}$$

Please cite this article as: Westerlund T., Lundell A. and Westerlund J., (2011), Some notes on convex relaxations, AIDIC Conference Series, 10, 383-392 DOI: 10.3303/ACOS1110042

where f is a convex objective function, g_m are functions defining inequality constraints, M is the number of inequality constraints and $\boldsymbol{x}$ is a vector of variables in X, a convex subset of R^N. Convex constraints can be included in g, but in this case only the nonconvex constraints need to be relaxed. An attractive convex relaxation of problem (1) is obtained by replacing the nonconvex functions with their tightest convex relaxations, *i.e.*, the convex envelopes conv g. This does not, however, necessarily result in the tightest possible convex relaxation of the feasible region of the optimization problem.

3. RELAXATIONS OF FUNCTIONS DEFINING INEQUALITY CONSTRAINTS

According to Tuy (1998): "A nonconvex inequality constraint $g(\boldsymbol{x}) \le 0, \boldsymbol{x} \in X$, where X is a convex set in R^n, can often be handled by replacing it with a convex inequality constraint $c(\boldsymbol{x}) \le 0$, where $c(\boldsymbol{x})$ is a convex minorant of $g(\boldsymbol{x})$ on X. The latter inequality is then called a convex relaxation of the former. Of course, the tightest relaxation is obtained when $c(\boldsymbol{x})$ = conv $g(\boldsymbol{x})$, the convex envelope, *i.e.*, the largest convex minorant, of $g(\boldsymbol{x})$." It should, however, be observed that conv g is the tightest relaxation of the function g over the convex set X (Sherali and Alameddine, 1990), and not the tightest convex relaxation of the set $\{\boldsymbol{x} \in X \mid g(\boldsymbol{x}) \le 0\}$ itself.

From an optimization point-of-view, the convex envelope of the feasible region is even more important than convex envelopes of the constraint functions, since the tightest convex relaxation of the feasible region is not generally obtained by replacing the functions in problem (1) by their convex envelopes. Instead, the tightest convex relaxation of the problem is given by the convex envelope of the set defining its feasible region, and the convex envelope of this set is the border of its convex hull.

4. CONVEX RELAXATION OF A FEASIBLE REGION AND ITS CONVEX ENVELOPE

Problem (1) is defined by a convex objective function, variables connected to the set X and a number of nonconvex inequality constraints. The level sets of the nonconvex inequality constraints can be defined as

$$L_\alpha^g = \bigcap_m \{\boldsymbol{x} \in X \mid g_m(\boldsymbol{x}) \le \alpha\}. \tag{2}$$

The feasible region of problem (1) can, thus, be defined as the level set $L_{\alpha=0}^g$, defined by the RHS = 0 of the inequality constraints. Now, observe that a potentially good convex relaxation of the feasible region of the problem can be obtained if the feasible region $L_{\alpha=0}^g$ of the nonconvex problem is replaced with $L_{\alpha=0}^{\mathrm{conv}\,g}$, *i.e.,* by replacing the nonconvex functions g by their convex envelopes conv g. However, there is no guarantee that $L_{\alpha=0}^{\mathrm{conv}\,g}$, will result in the tightest convex relaxation of $L_{\alpha=0}^g$ and we will later on illustrate, that this is not generally the case. Thus, replacing the nonconvex functions defining the inequality constraints with their convex envelopes does not necessarily result in the convex hull $\mathrm{conv}\,L_{\alpha=0}^g$, *i.e.*, the tightest convex relaxation, of the set $L_{\alpha=0}^g$. As mentioned previously, the convex envelope of a set is the border of its convex hull. Therefore, if the border of $\mathrm{conv}\,L_{\alpha=0}^g$ can be defined by convex functions q, different from the convex envelopes conv g, then a tighter, or at least an equally tight, convex relaxation of the feasible region of the problem will be obtained.

If the convex functions q over X are defined such that $q(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \text{conv}\, L^{g}_{\alpha=0}$, and $q(\boldsymbol{x}) > 0, \forall \boldsymbol{x} \notin \text{conv}\, L^{g}_{\alpha=0}$, then $L^{g}_{\alpha=0} \subseteq L^{q}_{\alpha=0}$ and the convex functions q giving the tightest convex relaxation of $L^{g}_{\alpha=0}$ are obtained when $L^{q}_{\alpha=0} = \text{conv}\, L^{g}_{\alpha=0}$. This results in $L^{g}_{\alpha=0} \subseteq L^{q}_{\alpha=0} \subseteq L^{\text{conv}\, g}_{\alpha=0}$.

Unfortunately, we do not have a general procedure to generate convex envelopes of the type q, defining the border of the convex hull of $\text{conv}\, L^{g}_{\alpha=0}$, for general classes of problems. We will, however, show by some examples that such functions exist and can be obtained.

5. NUMERICAL EXAMPLES

In the following, we will illustrate the issues discussed above with two numerical examples. In the first example $x \in R$. The second example is related to N-dimensional allocation where $\boldsymbol{x} \in R^{N}$.

5.1 Example 1 □ a univariate function

Consider an optimization problem of the type (1) with one nonconvex inequality constraint where the LHS of the inequality constraint is defined by the following function

$$g(x) = \frac{17}{3360}x^4 + \frac{107}{1120}x^3 - \frac{1073}{840}x^2 + \frac{807}{280}x + \frac{3}{2}, \qquad 0 \leq x \leq 7. \tag{3}$$

We will now illustrate different convex relaxations for this inequality constraint. The first relaxation is obtained by replacing the LHS, *i.e.*, the function $g(x)$, with its convex envelope. The convex envelope of $g(x)$ is given by

$$\text{conv}\, g(x) = \begin{cases} -0.488764x + 1.5 & \text{if } 0 \leq x \leq 4.8312, \\ g(x) & \text{if } 4.8312 < x \leq 7. \end{cases} \tag{4}$$

The function $g(x)$ in (3) is in this particular case an additive function of five terms: a constant term, three convex terms and one concave term (□ 1073/840 x^2). Taking the convex envelopes of each term separately, we obtain the following relaxation,

$$\text{approx conv}\, g(x) = g(x) + 1.2774x^2 - 1.2774 \cdot 7x, \qquad 0 \leq x \leq 7, \tag{5}$$

Finally, we will illustrate relaxations resulting in a feasible region $L^{q}_{\alpha=0}$ being exactly equal to that of the original problem, *i.e.*, $L^{g}_{\alpha=0}$. Two such relaxations are given: the first one is obtained by replacing the function $g(x)$ with the differentiable convex function

$$q_1(x) = \frac{5}{2}(x-4)(x-6) \tag{6}$$

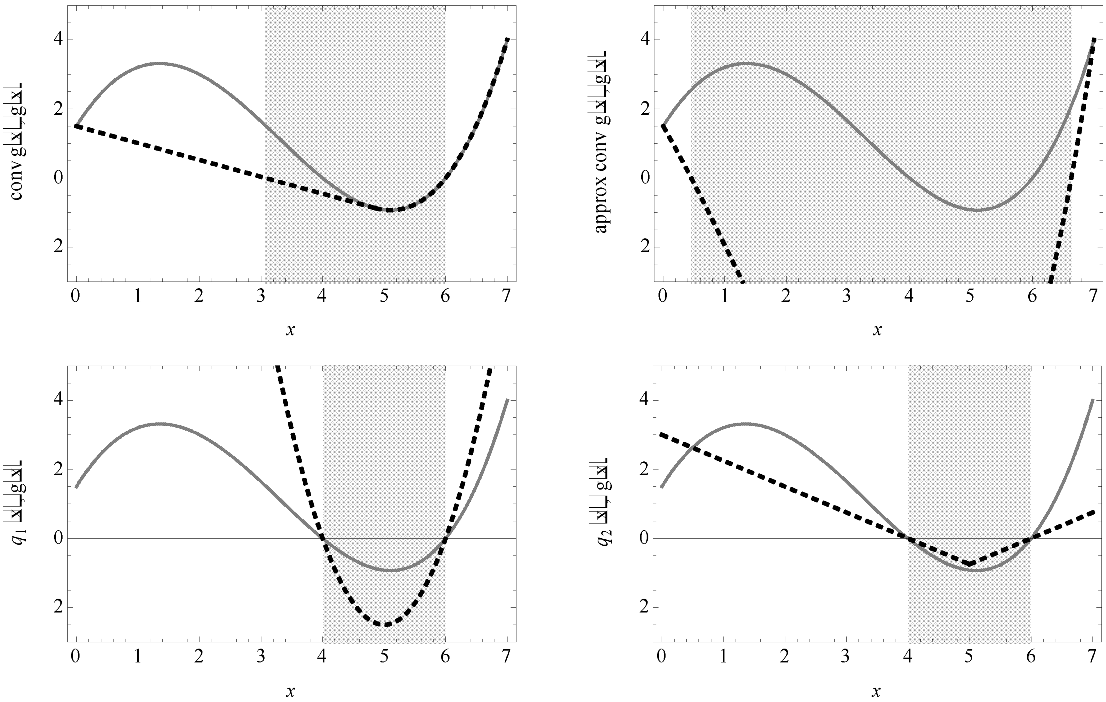

Fig. 1: Illustration of different convex relaxations for g(x) ≤ 0. The gray line is the plot of the function g(x), and the dashed black lines are the indicated convex underestimators. The gray areas indicate the overestimated feasible regions.

and the second one by replacing $g(x)$ with the function

$$q_2(x) = \max\left\{-\frac{3}{4}(x-4), \frac{3}{4}(x-6)\right\}. \tag{7}$$

The function $q_2(x)$ is the maximum of two linear functions, easily implemented in an optimization problem as two inequality constraints.

In Fig. 1, the function $g(x)$, as well as, the different convex underestimators are illustrated. From the figure, we find that the level sets for $g(x)$ at $\alpha = 0$ and the different convex relaxations are given by

$$L_{\alpha=0}^{g} = [4,6],\ L_{\alpha=0}^{\mathrm{conv}\, g} = [3.06896, 6],\ L_{\alpha=0}^{\mathrm{approx\ conv}\, g} = [0.45501, 6.63288],\ L_{\alpha=0}^{q_1} = [4,6], \text{ and } L_{\alpha=0}^{q_2} = [4,6].$$

Thus, we find that $L_{\alpha=0}^{g} = L_{\alpha=0}^{q_1} = L_{\alpha=0}^{q_2} \subset L_{\alpha=0}^{\mathrm{conv}\, g} \subset L_{\alpha=0}^{\mathrm{approx\ conv}\, g}$, and we conclude that $q_1(x)$ and $q_2(x)$ give the tightest convex relaxations. From the example above, we can further observe that valid linear cutting planes (Westerlund and Pörn, 2002) can in certain cases result in tighter convex relaxations of nonconvex constraints than achievable by replacing a constraint function by its convex envelope.

5.2 Example 2 – *N*-dimensional allocation

This example is related to a problem connected to N-dimensional allocation. A general model for this type of problem was presented in Westerlund et al. (2007). In the paper, items in an N-dimensional space were considered: rectangles in two dimensions, boxes in three dimensions, and so on. The sizes of the items were defined with fixed side lengths. In two related papers, Castillo et al. (2005) and Bonäs et al. (2007), the items were defined by their total areas, volumes etc. and aspect ratios were used to define maximum and minimum side lengths. The size constraint for an item in the N-dimensional case could thus generally be defined by a scalar parameter S and a product of variable side lengths x_i, restricted by defined aspect ratios in each direction in the N-dimensional space.

Thus, the size constraint for an item in an N-dimensional allocation problem can (together with overlapping protection constraints (Westerlund et al., 2007)), be defined as

$$\prod_{i=1}^{N} x_i \geq S. \tag{8}$$

The constraint function g, connected to an item, in an optimization problem of type (1), would then be given by

$$g(\boldsymbol{x}) = S - \prod_{i=1}^{N} x_i. \tag{9}$$

In a two-dimensional case, the constraint would contain a negative bilinear term and in the three-dimensional case a negative trilinear term, and so on. For such terms there are known convex envelopes, *cf.* McCormick (1976) and Al-Khayyal (1983) for bilinear term and Meyer and Floudas (2003) for trilinear terms. In this particular case $\boldsymbol{x} \in X \subset R_{+}^{N}$, and we observe that the function g is in fact quasiconvex. Since quasiconvex functions have convex level sets, the border of the convex hull of $L_{\alpha=0}^{g}$ must be given by the border of the level set $L_{\alpha=0}^{g}$ itself. The inequality constraint corresponding to expression (9) is written as

$$S - \prod_{i=1}^{N} x_i \leq 0. \tag{10}$$

This inequality can, however, be reformulated into convex form at the border (RHS = 0), for example, as follows

$$q_k(\boldsymbol{x}) = S \Big/ \prod_{\substack{i=1 \\ i \neq k}}^{N} x_i - x_k \leq 0, \quad k = 1,2,\ldots,N. \tag{11}$$

Since equation (9) and its convex reformulations in equation (11) result in identical solutions at RHS = 0, the expressions also exactly represent the same border of the level set $L_{\alpha=0}^{g}$. Thus, we may conclude that the nonconvex function g in an optimization problem of type (1) can be replaced with the convex constraints $q_k \leq 0$. In this particular case, we in fact obtain the border of the convex hull of the level set $L^{g}_{\alpha=0}$ (as it is identical to the level set itself) simply by replacing g in problem (1) with the convex functions q_k: $L^{g}_{\alpha=0} = L^{q}_{\alpha=0} \subseteq L^{\mathrm{conv}\,g}_{\alpha=0}$. When solving a problem of type (1) using g, or replacing g with conv g or q_k, we will thus obtain the minimum objective function value for the different domains as

$$f^*(\boldsymbol{x}^{\mathrm{conv}\,g}) \leq f^*(\boldsymbol{x}^{q}) = f^*(\boldsymbol{x}^{g}). \tag{12}$$

Note however that $q_k(\boldsymbol{x}) \leq \mathrm{conv}\ g(\boldsymbol{x}) \leq g(\boldsymbol{x})$ does not hold true in this case for all $\boldsymbol{x} \in X$. Consequently the functions q_k are neither convex minorants nor the convex envelope of the function g in X. Replacing g with conv g in problem (1) will thus not result in the tightest convex relaxation of the problem as mentioned in Tuy (1998). Instead, a tighter convex relaxation of the set $L^{g}_{\alpha=0}$, is obtained by replacing g in the problem with any of the functions q_k (resulting in the convex envelope of the set $L^{g}_{\alpha=0}$ for all $\boldsymbol{x} \in X$).

In fact we have, in this particular case, that

$$q_k(\boldsymbol{x}) = g(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in X : q_k(\boldsymbol{x}) = g(\boldsymbol{x}) = 0 \tag{13}$$

and

$$\begin{cases} q_k(\boldsymbol{x}) \wedge g(\boldsymbol{x}) > 0 & \forall \boldsymbol{x} \in \mathrm{X} : \prod_{\mathrm{i}=1}^{\mathrm{N}} x_i < S, \\ q_k(\boldsymbol{x}) \wedge g(\boldsymbol{x}) < 0 & \forall \boldsymbol{x} \in \mathrm{X} : \prod_{\mathrm{i}=1}^{\mathrm{N}} x_i > S. \end{cases} \tag{14}$$

Furthermore, if $g(\boldsymbol{x}) > 0$ then

$$\begin{cases} q_k(\boldsymbol{x}) < g(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \left\{ \mathrm{X} \mid \prod_{\mathrm{i}=1,\, \mathrm{i}\neq\mathrm{k}}^{\mathrm{N}} x_i > 1 \right\}, \\ q_k(\boldsymbol{x}) > g(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \left\{ \mathrm{X} \mid \prod_{\mathrm{i}=1,\, \mathrm{i}\neq\mathrm{k}}^{\mathrm{N}} x_i < 1 \right\}. \end{cases} \tag{15}$$

The conditions (14) and (15) can simply be obtained by inserting the expressions for the functions g and q_k, as in equations (9) and (11), into the inequalities. Similar conditions as in equations (15) can also be derived for the case when $g(\boldsymbol{x}) < 0$, the inequalities within the set braces in equation (15) only change direction. Note, that the conditions (14) and (15) do not imply that $q_k(\boldsymbol{x}) \geq g(\boldsymbol{x}) < 0$ or $q_k(\boldsymbol{x}) > g(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in X$. In fact $q_k(\boldsymbol{x}) > 0$, if $g(\boldsymbol{x}) > 0$ and $q_k(\boldsymbol{x}) < 0$ if $g(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \in X$ as indicated in equation (14).

Considering conv $g(\boldsymbol{x}) \geq q_k(\boldsymbol{x})$, we further find that this inequality is true when $q_k(\boldsymbol{x}) \geq g(\boldsymbol{x})$ as conv $g(\boldsymbol{x}) \leq g(\boldsymbol{x})$. Thus, replacing g with conv g in problem (1) will not result in the tightest convex relaxation of the problem. Instead, a tighter convex relaxation of the set $L^{g}_{\alpha=0}$, and thus, problem (1), is obtained by replacing g in the problem with q_k.

We will finally give an illustrative numerical example. Consider the special case where N = 2 and S = 50. The function g is then given by

$$g(\boldsymbol{x}) = 50 - x_1 x_2. \tag{16}$$

Convex envelopes of bilinear terms are given in McCormick (1976). Using the convex envelope of negative bilinear terms, the convex envelope of the function $g(\boldsymbol{x})$, where the bounds on the variables are, *e.g.*, $0.5 \leq x_1, x_2 \leq 10$, is given by

$$\operatorname{conv} g(\boldsymbol{x}) = 50 + \max\{-10x_1 - 0.5x_2 + 5, -0.5x_1 - 10x_2 + 5\}. \tag{17}$$

Furthermore, the functions q_k are given by

$$q_1(\boldsymbol{x}) = {}^{50}\!/_{x_2} - x_1 \quad \text{and} \quad q_2(\boldsymbol{x}) = {}^{50}\!/_{x_1} - x_2. \tag{18}$$

To exemplify the above discussion, let us consider the four points given in Table 1, which indicate the underestimation (or rather lack thereof) of the different formulations. In Fig. 2, the smaller figure shown in the upper left corner illustrates the level curves of the original nonconvex constraint. In the figure, the feasible region $L^{g}_{\alpha=0}$ for problem (1) with X = [0.5, 10] × [0.5, 10] and equation (16) is illustrated by a dark gray region. The figure in the upper right corner illustrates the level curves when the constraint function g has been replaced by its convex envelope. The light gray region is the relaxed convex feasible region $L^{\operatorname{conv} g}_{\alpha=0}$ obtained in this case. Finally the lower left and right figures illustrate the level curves when replacing the constraint function with the convex relaxations q_1 and q_2 respectively. The light gray regions illustrate the convex feasible regions $L^{q_1}_{\alpha=0}$ and $L^{q_2}_{\alpha=0}$. In Fig. 3, a plot of the constraint function g and the convex relaxations using q_1 and q_2 are illustrated. The four points in Table 1 are also included in the figure. The points are plotted for the original function g to simplify the identification of the original constraint and to differentiate the convex relaxations q_1 and q_2. Finally, a plane at zero is included. Note that the convex relaxations q_1 and q_2 approximates the original feasible region $L^{g}_{\alpha=0}$ in an exact way and do not cut of any part of the feasible region.

In Fig. 4, to the left the difference conv $g(\boldsymbol{x}) - g(\boldsymbol{x})$ is illustrated and to the right the difference $\max\{q_1(\boldsymbol{x}), q_2(\boldsymbol{x})\} - g(\boldsymbol{x})$. We observe that the convex envelope of the constraint function underestimates the function g in the entire region, even in the feasible region. However, the functions q_1 and q_2 underestimates the constraint function g only in a part of the infeasible region as the requirement for these functions is only to be positive and convex in this region. Furthermore, we find that $\max\{q_1(\boldsymbol{x}), q_2(\boldsymbol{x})\} - g(\boldsymbol{x}) = 0$ when $g(\boldsymbol{x}) = 0$ and that the functions q_1 and q_2 overestimates the function g in the feasible region. The overestimation is, however, not a flaw since the functions q_1 and q_2 need only to be negative and convex in this region.

Table 1: The function values for the points considered in Example 2

x_1	x_2	$g(\boldsymbol{x})$	conv $g(\boldsymbol{x})$	$q_1(\boldsymbol{x})$	$q_2(\boldsymbol{x})$
2	5	40	32.5	8	20
4	0.5	48	48	96	12
8	6.25	0	-11.5	0	0
10	10	-50	-50	-5	-5

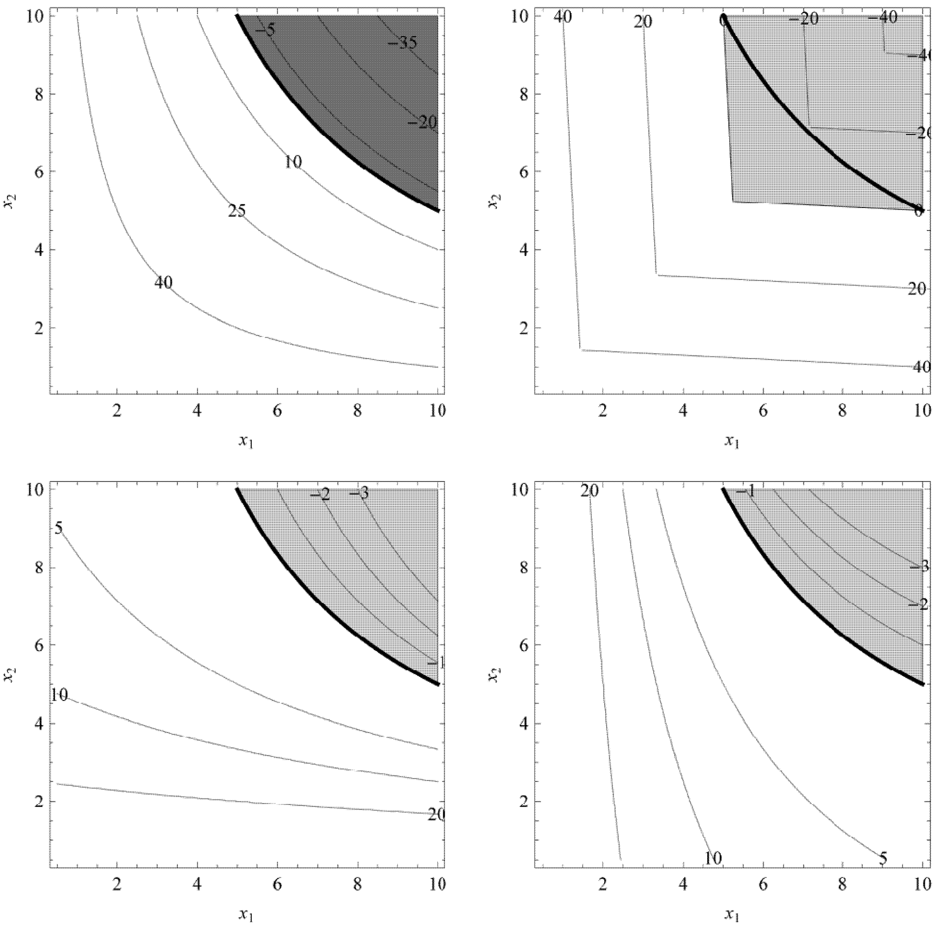

Fig. 2: Level curves and feasible regions indicated for g (upper left), conv *g (upper right),* q_1 *(lower left) and* q_2 *(lower right)*

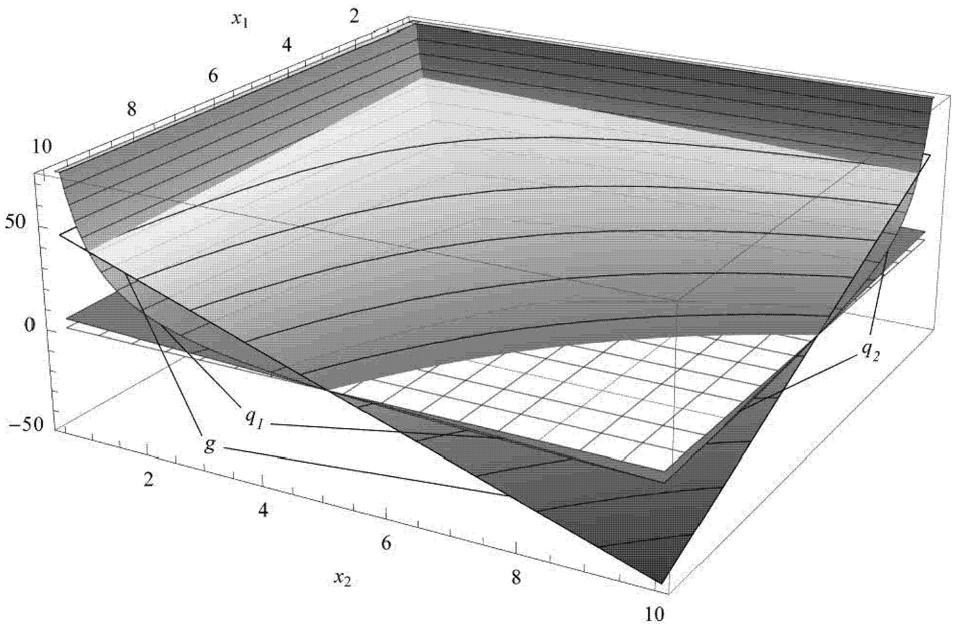

Fig. 3: The nonconvex function g, as well as the convex functions q_1 and q_2 from Ex. 2.

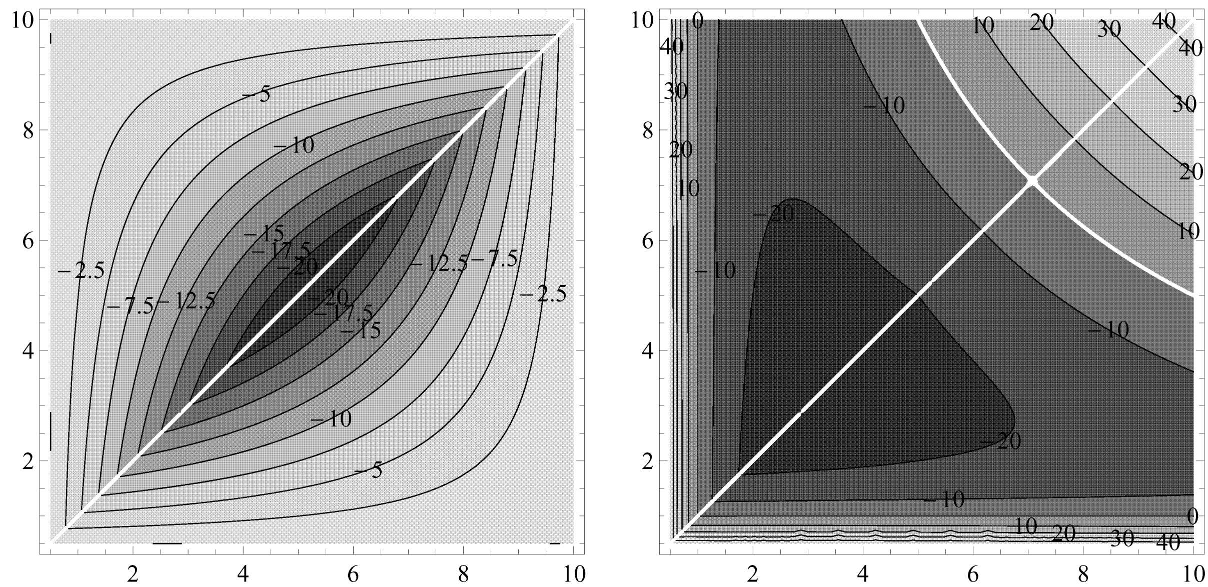

Fig. 4: Illustrations of conv $g(\boldsymbol{x}) \leq g(\boldsymbol{x})$ *(left) and* $\max\{q_1(\boldsymbol{x}), q_2(\boldsymbol{x})\} \leq g(\boldsymbol{x})$ *(right)*

6. CONCLUSIONS

In this paper, we have considered some issues related to convex relaxation techniques in constrained nonconvex optimization. We have pointed out the importance of differentiating between convex envelopes for functions and convex envelopes for sets when creating the tightest possible convex relaxations in constrained problems. The tightest convex relaxation of $L = \{\boldsymbol{x} \mid g_m(\boldsymbol{x}) \leq 0, m = 1,\ldots,M\}$ is conv L (*i.e.*, the convex hull of L) and is generally not obtained when the functions g_m are replaced by conv g_m (*i.e.,* the convex envelopes of the functions g_m). If the functions g_m are replaced by convex functions q_m defined by $q_m(\boldsymbol{x}) \leq 0 \, \forall \boldsymbol{x} \in \text{conv } L$ and $q_m(\boldsymbol{x}) > 0$ $\forall \boldsymbol{x} \notin \text{conv } L$, a tighter or equal tight convex relaxation of L is obtained than when replacing g_m with conv g_m. This was illustrated using two examples.

7. ACKNOWLEDGEMENTS

Financial support from the Foundation of Åbo Akademi University, as part of the grant for the Center of Excellence in Optimization and Systems Engineering, is gratefully acknowledged.

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